Strategy-proof Provision of Two Public Goods: the Lexmax Extension*

Lars Ehlers[†] and Sidartha Gordon[‡]

May 2011

Abstract

This paper studies the problem of providing two public goods for agents with single-peaked preferences. A decision rule selects two points on the segment $[0, 1]$ for the public goods for every profile of reported preferences. Agents compare public good pairs by the lexmax ordering over pairs induced by their single-peaked preference over single locations. We derive implications of *strategy-proofness* in this setting and compare them with those in the model with one public good and in the model with two public goods under the max extension. We characterize the class of decision rules satisfying *strategy-proofness*, *anonymity* and *continuity with respect to preferences*. We also characterize subclasses of rules that satisfy additional properties.

JEL classification: D63, D71, D81

Keywords: Single-peaked preferences, public goods, strategy-proofness.

*This research is financially supported by FQRSC (Québec). We thank Salvador Barberà and John Duggan for their insights and audiences at Universitat Autonoma de Barcelona, the Meetings of the Society for Social Choice and Welfare (2008), the workshop on Prior-free Mechanism Design (2009) and the workshop on Social Decisions (2010) for comments and questions.

[†]Université de Montréal and CIREQ.

[‡]Université de Montréal and CIREQ.

1 Introduction

We consider the problem of selecting two alternatives for the provision of two identical public goods. Agents have single-peaked preferences on a closed interval. Preferences are single-peaked if up to a certain point, the peak, preferences are strictly increasing, and strictly decreasing beyond that point. The feasible set of alternatives consists of pairs providing for each public good a level. Agents compare alternatives by the lexmax ordering over these pairs induced by their single-peaked preference. More explicitly, when comparing two pairs, an agent first compares his preferred level in both pairs. Only if he is indifferent between his preferred level in both pairs, he then compares the other two levels. A public good economy is completely described by a list of parameters such as the set of the agents and their preferences over the closed interval. A decision rule is a systematic way to assign the levels of the two public goods for each economy, that is for every profile of reported preferences.

An important application of the model is the problem of locating two pure public good facilities, such as electricity generating plants, or telecommunication repeaters. The lexmax ordering over pairs captures the fact that each agent will only use his most preferred alternative in the provided pair. Thus, his first order preference is determined by comparing the best alternatives in each pair. He will use the second option only as a plan B, in case of a breakdown of the first best option.[1]

We are interested in rules that satisfy three main properties. *Strategy-proofness* requires that no agent can individually benefit by misreporting his preferences. *Anonymity* says that the selection does not depend on agents' labels. Finally we impose *continuity* in the agent's preferences. Our main contribution is the introduction of a new class of rules for two public goods, the class of *delta rules*, and its characterization. Our main result is that the three above properties characterize the class of *delta rules*. We also characterize subclasses of rules that (separately) satisfy additional requirements, such as *group strategy-proofness*, *unanimity*, *Pareto-optimality*, *peaks-selection*, *diversity*, *population-monotonicity* and *replacement-domination*.

The organization of the paper is as follows. In Section 2, we discuss related

[1] Most of the electricity consumed in Barcelona comes from a plant located in Girona. If this plant breaks down, the electricity is brought from a plant located in Grenoble. We thank Salvador Barberà for this example.

literature. Section 3 presents the model. In Section 4, we introduce the class of *delta rules* and characterize them as the rules that satisfy *strategy-proofness*, *anonymity* and *continuity*. In Section 5, we characterize interesting subclasses of rules that satisfy additional requirements. In Section 6, we investigate the implications of *peaks-selection*. In Section 7, we investigate upon the existence of a *Condorcet winner*.

2 Related literature

2.1 One good

Black (1948) introduces the problem of selecting the level of a single public good, when agents have single-peaked preferences. Moulin (1980), introduce the generalized median rules and show that these rules are the only ones that satisfy *strategy-proofness*, *anonymity* and *continuity* in the one good model.[2] With n agents, such a rule is described by $n+1$ fixed cardinal points. The rule selects the median of the n agents peaks and the $n+1$ cardinal points.

Our *delta rules* can be viewed as an adaptation of the generalized median rules for the problem of selecting multiple public good levels, under the lexmax extension. A *delta rule* essentially locates each public good according to a generalized median rule, whose cardinal points are not fixed, but depend in a particular way (which we describe) on the entire preference profile. In particular, a double generalized median rule, that selects each good according to a generalized median rule with fixed cardinal points is a special instance of a *delta rule*.

2.2 Multiple public goods: the max extension

The axiomatic study of the provision of multiple public goods is initiated by Miyagawa (1998, 2001). Miyagawa studies a model where alternatives are compared according to the max extension: a pair (or tuple) is preferred to another pair if and only if the best element of the first pair is preferred to the best element of the second pair.[3]

[2] See also Barberà and Jackson (1994) and Ching (1997).

[3] The implication of several axioms have been studied in this model. Miyagawa (2001) studies *replacement-domination*. Miyagawa (1998) studies *population-monotonicity*. Barberà and Beviá (2002, 2005) and Ju (2008) study *consistency*. Bochet and Gordon (2010) and Bochet, Gordon and

Among other questions, Miyagawa (1998) examines the implications of *strategy-proofness* in this model. He shows in particular that the only rule that satisfies *Pareto-optimality*, *strategy-proofness* and *continuity* is the extreme-peaks rule, which selects the lowest peak and the highest peak for any preference profile. Gordon and Bochet (2010) characterize a class of rules that allocate public goods on agents' peaks according to a priority ordering over unanimous profiles and satisfy *strategy-proofness*, the hierarchical rules.

The closest paper to ours in the max extension literature is by Heo (2011). Heo defines, for a given pair of public good levels and a given preference profile, the users of a good as the agents who prefer it to the other and, possibly, the agents who are indifferent between this good and the other. She then introduces a new property, *users-only*, which says that the level of each good should only depend on the preferences of its own users. Heo (2011) shows than the double generalized medians rules, that allocate each good according to a generalized median rule with fixed cardinal points are the only rules that satisfy *strategy-proofness*, *continuity*, *anonymity* and *users-only*.

The class characterized by Heo (2011) is a subclass of ours. However, her result is not logically implied by ours, because *strategy-proofness* is a stronger requirement under the lexmax than under the max extension. In particular the set of rules that satisfy our three main properties (*strategy-proofness*, *continuity*, and *anonymity*) in the max extension model contains the *delta rules*, but also other complicated ones, which do not depend only on the agents' peaks, but also on agents' non-peak preference information. Heo (2011) also considers *group strategy-proofness*, which also has different implications in the two models. While only some of the rules characterized by Heo (2011) are *group strategy-proof* with the max extension, all *delta rules* satisfy this requirement under the lexmax extension.

2.3 Multiple public goods: the lexmax extension

Ehlers (2002, 2003) introduces the lexmax extension in the study of the provision of multiple public goods. He focuses on solidarity axioms and identifies and characterizes two interesting classes of rules, the single-plateau rules and the single-peaked rules.

Saran (2009) study *object-population monotonicity*.

Both of these classes are subsets of the class of *delta rules*.[4] We discuss the exact relation between Elhers' (2002, 2003) result and ours in Section 5.

2.4 Other public good models with single-peaked preferences

Following Moulin (1980), a large literature studies the implications of *strategy-proofness* in more general, sometimes abstract, common domains of single-peaked preferences. It shows that Moulin's (1980) generalized median voter rules can be adapted for these settings as well and characterized by *strategy-proofness* and other properties. For example, Border and Jordan (1983) study the location of a public good on an Euclidean space. Barberà, Sonnenschein and Zhou (1990) study the selection of a subset from a set of candidates. Elhers, Peters and Storcken (2002) study probabilistic decision rules. Schummer and Vohra (2002) studies the location of one public good on a network. For a survey of this large literature, see Barberà (2008).

Our model departs from the models in this literature in two ways. The first one is that these models usually consider continuous preferences (sometimes by assuming that the set of alternatives is finite). This is not the case here, as lexmax preferences are never continuous. The second difference is crucial. These models generally assume that the common domain satisfies a richness condition. In particular, they assume that each feasible alternative is the preferred alternative of some preference in the common domain. This is not the case here, since only alternatives on the diagonal $(x, x) \in [0, 1]^2$ are the preferred alternative of some preference in the domain, even though all alternatives in $[0, 1]^2$ are feasible. The absence of the richness condition allows for a large set of *strategy-proof* rules.

3 The Model

Let $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$, be the set of agents. Each agent $i \in N$ is equipped with a *single-peaked* continuous preference relation R_i defined over $[0, 1]$. We denote the associated strict relation by P_i and the indifference relation by I_i. *Single-peakedness*

[4] Gordon (2007) has shown that in any pure public good model, of which the provision of multiple public goods is a special case, under Pareto-efficicency, population-monotonicity implies strategy-proofness. This explains the inclusion of the class described by Ehlers (2003) into ours.

means that there exists a point $p(R_i) \in [0,1]$, called the *peak* of R_i, such that for all $x, y \in [0,1]$, if $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, then $y \; P_i \; x$. By $\mathcal{R}$ we denote the class of all single-peaked preferences over $[0,1]$. By $\mathcal{R}^N$ we denote the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. For $S \subseteq N$ the restriction $(R_i)_{i \in S}$ of R to S is denoted by R_S. Given a profile $R \in \mathcal{R}^N$, the smallest peak is denoted by $\underline{p}(R)$, and the largest peak by $\overline{p}(R)$. For $S \subseteq N$, two profiles $R, R' \in \mathcal{R}^N$ are called S-deviations if $R_{N \setminus S} = R'_{N \setminus S}$. For $i \in N$, two profiles are i-deviations if they are $\{i\}$-deviations.

Two public goods have to be selected. An *alternative* is a pair (x, y) such that $0 \leq x \leq y \leq 1$. The set of alternatives is denoted by $[0,1]^2$.

A *decision rule* is a mapping φ associating to every profile $R \in \mathcal{R}^N$ an alternative. We write $\varphi(R) = (\varphi_1(R), \varphi_2(R))$.

Preferences are extended over the set of alternatives $[0,1]^2$. With the interpretation that each alternative is an option set, we assume that each agent compares two alternatives via the *lexmax ordering* induced by his single-peaked preference. Abusing notation, we use the same symbols to denote preferences over alternatives. Thus, for two pairs (x, y), (x, y) and two permutations τ of $\{x, y\}$, and τ' of $\{x', y'\}$ such that $\tau(x) \; R_i \; \tau(y)$ and $\tau'(x') \; R_i \; \tau'(y')$ we have $(x, y) \; P_i \; (x', y')$ if either $\tau(x) \; P_i \; \tau'(x')$ or if $\tau(x) \; I_i \; \tau'(x')$ and $\tau(y) \; P_i \; \tau'(y')$. If $\tau(x) \; I_i \; \tau'(x')$ and $\tau(y) \; I_i \; \tau'(y')$, then $(x, y) \; I_i \; (x', y')$.

4 Strategy-proofness, anonymity and continuity: a characterization

In this section, we first introduce our main properties, *strategy-proofness*, *anonymity* and *continuity*. Second, we characterize the class of one agent decision rules that satisfy *strategy-proofness* and *continuity*. Third, we turn to the n agent problem. We introduce the *delta rules* and show that this class is characterized by our three main three properties.

4.1 Main properties

First we introduce our main properties. *Strategy-proofness* means that an agent does not gain by misreporting his true preference.

Strategy-proofness: For all $i \in N$ and all i-deviations $R, R' \in \mathcal{R}^N$, $\varphi(R)\ R_i\ \varphi(R')$.

Remark 1 *Note that strategy-proofness with respect to the lexmax extension implies strategy-proofness with respect to the max extension, studied by Miyagawa (1998) and Heo (2011).*

Anonymity means that the decision rule is symmetric in its arguments.

Anonymity: For all permutations σ of N, $\varphi(R) = \varphi(\sigma(R))$.[5]

To introduce *continuity*, we need to first define an ordinal metric on $\mathcal{R}$. A preference R_i is uniquely represented by a function $r_i : [0,1] \rightarrow [0,1]$, such that for all $x \in [0,1]$, if $x = p(R_i)$ then $r_i(x) = x$. Otherwise, $r_i(x)$ is the (necessarily unique) location such that $r_i(x)\ I_i\ x$ and $r_i(x) \neq x$, if such a location exists. If no such location exists, let $r_i(x) = 0$ if $p(R_i) < x$ and let $r_i(x) = 1$ if $x < p(R_i)$. Let the metric on $\mathcal{R}$ be such that

$$d(R_i, R_i') := \sup_{[0,1]} |r_i(x) - r_i'(x)|.$$

Continuity: A rule satisfies this property if the function φ is continuous for the metric $d(\cdot,\cdot)$ on $\mathcal{R}$.

4.2 One agent

As in Moulin (1980) we first characterize the decision rules satisfying *strategy-proofness* and *continuity* for one agent. This result will be used in further subsections.

Recall that a one-person decision rule $\varphi : \mathcal{R} \rightarrow [0,1]$ for one public good satisfies *strategy-proofness* and *continuity* if and only if there exist $\underline{a}, \overline{a} \in [0,1]$ such that

$$\varphi(R_1) = med(\underline{a}, p(R_1), \overline{a}) \text{ for all } R_1 \in \mathcal{R}.$$

[5] As usual, $\sigma(R)$ is the permuted profile R according to σ.

Back to the two good problem, let $[\underline{a}, \overline{a}], [\underline{b}, \overline{b}] \subseteq [0,1]$ be such that $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$. The range of the smallest public good will be $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ will be the range of the greatest public good. A pair of functions

$$b : [\underline{a}, \overline{a}] \longmapsto [\underline{b}, \overline{b}]$$
$$a : [\underline{b}, \overline{b}] \longmapsto [\underline{a}, \overline{a}]$$

is called feasible if

- a and b are continuous,
- for all $x \in [\underline{a}, \overline{a}] \cap [\underline{b}, \overline{b}]$, $a(x) = x = b(x)$,
- $a(\underline{b}) = \min\{\overline{a}, \underline{b}\}$ and $b(\overline{a}) = \max\{\overline{a}, \underline{b}\}$,
- for all $x \in [\underline{a}, \overline{a}]$, $x \leq a(b(x))$,
- for all $x \in [\underline{b}, \overline{b}]$, $b(a(x)) \leq x$.

These restrictions are illustrated in Figures 1 and 2. Denote by $\mathcal{F}$ the set of all feasible pairs (a, b). For $(a, b) \in \mathcal{F}$ we define the one-person decision rule $\varphi^{(a,b)}$ based on (a, b) as follows:

$$\varphi_1^{(a,b)}(R_1) = med(\underline{a}, p(R_1), a(med(\underline{b}, p(R_1), \overline{b})))$$

and

$$\varphi_2^{(a,b)}(R_1) = med(b(med(\underline{a}, p(R_1), \overline{a})), p(R_1), \overline{b}).$$

Observe that for the smallest public good, we have $\varphi_1^{(a,b)}(R_1) = med(\underline{a}, p(R_1), a(\underline{b}))$ whenever $p(R_1) < \underline{b}$. This is just as in Moulin (1980) where $\underline{a}$ and $a(\underline{b})$ play the role of cardinal points. Only when $p(R_1)$ is large enough, or more precisely when $p(R_1)$ is entering the range of the greatest public good, then $p(R_1)$ is translated by the function a and we have $\varphi_1^{(a,b)}(R_1) = a\,(p(R_1))$ whenever $p(R_1) \in [\underline{b}, \overline{b}]$. If $p(R_1) > \overline{b}$, then $\varphi_1^{(a,b)}(R_1) = a(\overline{b})$.

An analogous description applies to the largest public good.

By Theorem 1, these decision rules are characterized by *strategy-proofness* and *continuity*.

Theorem 1 *Let φ be a one-person decision rule. Then, φ satisfies strategy-proofness and continuity if and only if for some $(a,b) \in \mathcal{F}$, $\varphi = \varphi^{(a,b)}$.*

We first show that if φ satisfies *strategy-proofness* and *continuity*, then the selected levels only depend on the peaks of the agents in the profile.

Peaks-onliness: For all $R, R' \in \mathcal{R}^N$, if for all $i \in N$, $p(R_i) = p(R'_i)$, then $\varphi(R) = \varphi(R')$.

Lemma 1 *Let φ be a one-person decision rule that satisfies strategy-proofness and continuity. Then φ satisfies peaks-onliness.*

Proof. Let $R \in \mathcal{R}$, $i \in N$ and $R'_i \in \mathcal{R}$ such that $p(R'_i) = p(R_i)$. We will show that $\varphi(R) = \varphi(R'_i, R_{-i})$. Let $\Omega := [\underline{\omega}, \overline{\omega}]$, with $\underline{\omega} < \overline{\omega}$. Let $R^*_i : \Omega \to \mathcal{R}$ be a continuous path such that $R^*_i(\underline{\omega}) = R_i$ and $R^*_i(\overline{\omega}) = R'_i$. Let $x(\omega)$ and $y(\omega)$ be two functions such that $\varphi(R^*_i(\omega), R_{-i}) = (x(\omega), y(\omega))$ for all $\omega \in \Omega$. By *continuity*, the functions $x(\cdot)$ and $y(\cdot)$ are also continuous.

Consider for sets of parameters on the path:

$$\Omega_1 := \{\omega \in [0,1] : p(R_i) \leq x(\omega) \leq y(\omega)\},$$
$$\Omega_2 := \{\omega \in [0,1] : x(\omega) \;\; R^*_i(\omega) \;\; y(\omega) \text{ and } x(\omega) \leq p(R_i) \leq y(\omega)\},$$
$$\Omega_3 := \{\omega \in [0,1] : y(\omega) \;\; R^*_i(\omega) \;\; x(\omega) \text{ and } x(\omega) \leq p(R_i) \leq y(\omega)\},$$
$$\Omega_4 := \{\omega \in [0,1] : x(\omega) \leq y(\omega) \leq p(R_i)\}.$$

Each of these sets is closed. Then for all $\omega \in \Omega_1$, the preference $R^*_i(\omega)$ induces the same linear ordering over the pairs in the set $\{(x(\omega'), y(\omega')) : \omega' \in \Omega_1\}$. A pair is preferred to another if either its first coordinate is lower, or the first coordinates are equal but the its second coordinate is lower. Let (x^1, y^1) be the top element of the set $\{(x(\omega'), y(\omega')) : \omega' \in \Omega_1\}$ for any preference $R^*_i(\omega)$ with $\omega \in \Omega_1$. By *strategy-proofness*, we have $(x(\omega), y(\omega)) = (x^1, y^1)$ for all $\omega \in \Omega_1$, i.e. the function $(x(\cdot), y(\cdot))$ is constant on Ω_1. Similarly, this function is constant on Ω_2, Ω_3 and Ω_4. Since $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, it follows that the image of the function $(x(\cdot), y(\cdot))$ has at most four elements. Since this function is continuous, it must be constant. In particular, $\varphi(R) = \varphi(R'_i, R_{-i})$. *Peaks-onliness* follows then directly from this property.■

This implication holds in a large class of domains, as shown by Weymark (2008). Note however that our model does not satisfy the richness condition of the domain and the continuity or preferences under which he obtains this result. Interestingly, in the max extension model, this implication does not hold but Heo (2011) shows that *strategy-proofness*, *continuity* and *users-only* imply *peaks-only* in that model. We are now ready to prove Theorem 1.

Proof of Theorem 1. (Only if) Let φ satisfy *strategy-proofness* and *continuity*. Define for all $R_1 \in \mathcal{R}$, $\varphi(R_1) := \hat{\varphi}(p(R_1))$. Note that by *peaks-onliness* of φ, $\hat{\varphi} : [0,1] \to [0,1]^{\{1,2\}}$ is well-defined. Obviously $\hat{\varphi}$ inherits *continuity* from φ.

Let

$$\underline{a} := \min_{x\in[0,1]} \hat{\varphi}_1(x),\ \overline{a} := \max_{x\in[0,1]} \hat{\varphi}_1(x),\ \underline{b} := \min_{x\in[0,1]} \hat{\varphi}_2(x),\ \text{and}\ \ \overline{b} := \max_{x\in[0,1]} \hat{\varphi}_2(x).$$

By *continuity* of φ, these numbers are well-defined.

First, let $[\underline{a},\overline{a}] \cap [\underline{b},\overline{b}] \neq \emptyset$, i.e. $\underline{b} \leq \overline{a}$. We show that for all $x \in [\underline{b},\overline{a}]$ we have $\hat{\varphi}_1(x) = \hat{\varphi}_2(x) = x$. By definition and *strategy-proofness*, $x \in \{\hat{\varphi}_1(x), \hat{\varphi}_2(x)\}$. Suppose $x = \hat{\varphi}_1(x)$ and $x < \hat{\varphi}_2(x)$. Since $x \in [\underline{b},\overline{b}]$, there exists some $y \in [0,1]$ such that $\hat{\varphi}_2(y) = x$ and $\hat{\varphi}_1(y) \leq x$. But then we may choose $R_1 \in \mathcal{R}$ such that $\hat{\varphi}_1(y) P_1 \hat{\varphi}_2(x)$ and $p(R_1) = x$. Now this contradicts *strategy-proofness* of φ because $\varphi(R_1) = \hat{\varphi}(x)$ and by reporting R_1' such that $p(R_1') = y$ we have $\varphi(R_1') = \hat{\varphi}(y)$. Hence, for all $x \in [\underline{a},\overline{a}] \cap [\underline{b},\overline{b}]$,

$$\hat{\varphi}_1(x) = \hat{\varphi}_2(x) = x. \tag{1}$$

Second, we define a and b. Let

$$a(x) := \hat{\varphi}_1(x) \text{ for all } x \in [\underline{b},\overline{b}].$$

and

$$b(x) := \hat{\varphi}_2(x) \text{ for all } x \in [\underline{a},\overline{a}],$$

Note that by definition of $\hat{\varphi}$, (1), and *strategy-proofness* of φ, for all $x \in [\underline{a},\overline{a}]$, $\hat{\varphi}_1(x) = x$, and for all $x \in [\underline{b},\overline{b}]$, $\hat{\varphi}_2(x) = x$. Thus, $a : [\underline{b},\overline{b}] \to [\underline{a},\overline{a}]$ and $b : [\underline{a},\overline{a}] \to [\underline{b},\overline{b}]$ are well-defined.

Third, we show that the pair (a,b) is feasible. By *continuity* of $\hat{\varphi}$, both a and b are continuous. By (1) and the definitions, for all $x \in [\underline{a},\overline{a}] \cap [\underline{b},\overline{b}]$, $b(x) = x = a(x)$.

Furthermore, $\overline{a} = \hat{\varphi}_1(\overline{a})$ and by definition $b(\overline{a}) \geq \max\{\overline{a}, \underline{b}\}$. If the inequality is strict, then by (1), $\overline{a} < \underline{b}$. Furthermore, by definition and *strategy-proofness*, $\hat{\varphi}_2(\underline{b}) = \underline{b}$ and $\hat{\varphi}_1(\underline{b}) < \overline{a}$. But now our definitions, $\overline{a} < \underline{b}$, the above facts and *strategy-proofness* imply $\hat{\varphi}(\frac{1}{2}(\overline{a} + \underline{b})) = (\overline{a}, \underline{b})$. This is a contradiction to *strategy-proofness* since for $R_1, R'_1 \in \mathcal{R}$ such that $p(R_1) = \overline{a}$ and $p(R'_1) = \frac{1}{2}(\overline{a} + \underline{b})$ we have $\varphi(R_1) = \hat{\varphi}(\overline{a})$ and $\varphi(R'_1) = \hat{\varphi}(\frac{1}{2}(\overline{a} + \underline{b}))$. By $\overline{b}\ P_1\ b(\overline{a})$, agent 1 profitably manipulates at R_1 via R'_1, a contradiction to *strategy-proofness*. Thus, $b(\overline{a}) = \max\{\overline{a}, \underline{b}\}$ and similarly, $a(\underline{b}) = \min\{\overline{a}, \underline{b}\}$.

By *strategy-proofness*, *peaks-onliness*, and the definitions, it is straightforward that for all $x \in [\underline{a}, \overline{a}]$, $x \leq a(b(x))$. Similarly, for all $x \in [\underline{b}, \overline{b}]$, $b(a(x)) \leq x$.

Fourth, we show that $\varphi = \varphi^{(a,b)}$. Let $R_1 \in \mathcal{R}$. If $p(R_1) < \underline{a}$, then by *strategy-proofness*, *continuity*, and *peaks-onliness*, $\varphi_1(R_1) = \hat{\varphi}_1(p(R_1)) = \underline{a} = \varphi_1^{(a,b)}(R_1)$. If $p(R_1) \in [\underline{a}, \overline{a}]$, then $\varphi_1(R_1) = p(R_1) = \varphi_1^{(a,b)}(R_1)$. If $p(R_1) \in]\overline{a}, \underline{b}[$, then by definition and *strategy-proofness*, $\varphi_1(R_1) = \overline{a}$. Since $a(\underline{b}) = \min\{\overline{a}, \underline{b}\}$, we also have $\varphi_1^{(a,b)}(R_1) = \overline{a}$. If $p(R_1) \in [\underline{b}, \overline{b}]$, then $\varphi_1(R_1) = a(p(R_1)) = \varphi_1^{(a,b)}(R_1)$. If $p(R_1) > \overline{b}$, then by *strategy-proofness*, *continuity*, and *peaks-onliness*, $\varphi_1(R_1) = \hat{\varphi}_1(p(R_1)) = a(\overline{b}) = \varphi_1^{(a,b)}(R_1)$. Hence, $\varphi_1(R_1) = \varphi_1^{(a,b)}(R_1)$. Similarly, it follows $\varphi_2(R_1) = \varphi_2^{(a,b)}(R_1)$, the desired conclusion.

(If) If for some $(a, b) \in \mathcal{F}$ we have $\varphi = \varphi^{(a,b)}$, then it is straightforward to verify that φ satisfies *strategy-proofness* and *continuity*. ■

The following example gives some insight to the extent of "arbitrariness" of feasible pairs of functions.

Example 1 *Let* $a(y) = \frac{1}{2}$ *for all* $y \in [\frac{1}{2}, 1]$ *and* $b : [0, \frac{1}{2}] \to [\frac{1}{2}, 1]$ *be an arbitrary continuous function such that* $b(\frac{1}{2}) = \frac{1}{2}$. *Then* $[\underline{a}, \overline{a}] = [0, \frac{1}{2}]$ *and* $[\underline{b}, \overline{b}] = [\frac{1}{2}, 1]$. *Obviously,* $b(\overline{a}) = \frac{1}{2} = \max\{\overline{a}, \underline{b}\}$ *and* $a(\underline{b}) = \frac{1}{2} = \min\{\overline{a}, \underline{b}\}$. *Furthermore, for all* $x \in [0, \frac{1}{2}]$, $x \leq \frac{1}{2} \leq a(b(x))$ *and (i) for* $x < \frac{1}{2}$, $x < \frac{1}{2} = a([\frac{1}{2}, b(x)[)$ *and (ii) for* $x = \frac{1}{2}$, $[\frac{1}{2}, b(\frac{1}{2})[= \emptyset$. *Finally, for all* $x \in [\frac{1}{2}, 1]$, *both* $b(a(x)) = b(\frac{1}{2}) = \frac{1}{2} \leq x$ *and* $b(]a(x), \frac{1}{2}]) = b(\emptyset) = \emptyset$. *Hence,* (a, b) *is feasible and* $\varphi^{(a,b)}$ *satisfies strategy-proofness and continuity.*

4.3 Many agents

We first introduce the class of n agents *delta rules.* We then show that our three main properties characterize it.

4.3.1 Delta rules

Let $\mathcal{F}_n$ be the set of lists (a, b, x, y) where $x = (x_{s,t})_{0\leq s\leq t\leq n}$ and $y = (y_{s,t})_{0\leq s\leq t\leq n}$ are elements of $[0,1]^{\{(s,t)\in\mathbb{N}^2:0\leq s\leq t\leq n\}}$ such that, for each (s,t) such that $s+t\leq n-1$,

$$(-x_{s,t}, y_{s,t}) \leq (-x_{s+1,t}, y_{s+1,t}) \tag{A1}$$

$$(-x_{s,t}, y_{s,t}) \leq (-x_{s,t+1}, y_{s,t+1}) \tag{A2}$$

and for each (s,t) such that $s\geq 1$, $t\leq n-1$, and $s+t\leq n$,

$$(x_{s,t}, y_{s,t}) \leq (x_{s-1,t+1}, y_{s-1,t+1})\,; \tag{A3}$$

and $a = (a_0, ..., a_{n-1})$ and $b = (b_0, ..., b_{n-1})$ are continuous functions

$$a_s : [y_{s,0}, y_{s,n-s}] \to [x_{n,0}, x_{s,0}]$$
$$b_t : [x_{n-t,t}, x_{0,t}] \to [y_{0,t}, y_{0,n}]\,.$$

that satisfy the following conditions:

For all (s,t) such that $s+t\leq n-1$:

$$a_s\left([y_{s,t}, y_{s,t+1}]\right) \subseteq \left[\min\left\{x_{s+1,t}, y_{s,t}\right\}, \min\left\{x_{s,t}, y_{s,t+1}\right\}\right]; \tag{B1}$$

$$b_t\left([x_{s+1,t}, x_{s,t}]\right) \subseteq \left[\max\left\{x_{s+1,t}, y_{s,t}, \right\}, \max\left\{x_{s,t}, y_{s,t+1}\right\}\right]; \tag{B2}$$

$$x_{s,t} \leq y_{s,t} \implies a_s\left(y_{s,t}\right) = x_{s,t} \text{ and } b_t\left(x_{s,t}\right) = y_{s,t}; \tag{B3}$$

$$\text{For all } x \in [x_{n-t,t}, x_{s,t}]\,,\ \ x \leq a_s\left(b_t\left(x\right)\right); \tag{B4}$$

$$\text{For all } y \in [y_{s,t}, y_{s,n-s}]\,,,\ \ y \geq b_t\left(a_s\left(y\right)\right). \tag{B5}$$

Conditions $(B4)$ and $(B5)$ are in fact equivalent. They say on the rectangle to the north-west of the pair $(x_{s,t}, y_{s,t})$, i.e.

$$\left\{(x,y) : (-x_{s,t}, y_{s,t}) \leq (-x, y) \leq (0,1)\right\},$$

the graphs of a_s is always to the north-east of the graph of b_t. The two graphs have no strict crossings in this rectangle, although they may touch without crossing (for example, they may be tangent). Moreover:

$$\text{For all } y \in [y_{s,0}, y_{s,n-s}] \cap \left[x_{s,0}, x_{s-1,n-(s-1)}\right], a_s(y) = y; \tag{C}$$
$$\text{For all } x \in [x_{n-t,t}, x_{0,t}] \cap \left[y_{0,t}, y_{n-(t-1),t-1}\right], b_t(x) = x;$$

$$\text{For all } y \in \left[y_{s+1,0}, y_{s+1,n-(s+1)}\right], a_{s+1}(y) \leq a_s(y); \tag{D}$$
$$\text{For all } x \in \left[x_{n-(t+1),t+1}, x_{0,t+1}\right], b_t(x) \leq b_{t+1}(x);$$

$$\text{For all } x \in [x_{s,n-s}, x_{s,0}] \cap b_{n-s}^{-1}([y_{s,0}, y_{s,n-s}]), x \geq a_s(b_{n-s}(x)); \tag{E}$$
$$\text{For all } y \in [y_{s,0}, y_{s,n-s}] \cap a_s^{-1}([x_{s,n-s}, x_{0,n-s}]), y \leq b_{n-s}(a_s(y)).$$

The two inequalities (E) are in fact equivalent. They complement the conditions $(B4)$ and $(B5)$. They say on the rectangle to the south-east of the pair $(x_{s,n-s}, y_{s,n-s})$, i.e.

$$\{(x,y) : (-1,0) \leq (-x,y) \leq (-x_{s,t}, y_{s,t})\},$$

the graph of a_s is always to the south-west relative to the graph of b_t. The two graphs have no strict crossings in this rectangle, although they may touch without crossing (for example, they may be tangent). Moreover, the graphs of the functions $a_s(\cdot)$ and $b_{n-s}(\cdot)$ do not (strictly) cross anywhere else in $[0,1]^2$ than at $(x_{s,t}, y_{s,t})$.

These restrictions are illustrated in Figure 3. For $(a,b,x,y) \in \mathcal{F}_n$ we define the *delta rule* $\varphi^{(a,b,x,y)}$ associated with (a,b,x,y) as follows. For all R_N, let $p(R_N) = (p_1, ..., p_n)$ be the vectors the peaks in $p(R_N)$ ranked in increasing order, so that $p_1 \leq ... \leq p_n$. The analogous of the cardinal points in the single public good case are here cardinal functions, defined as follows:

$$\alpha_0(p) := a_0(med(p_1, ..., p_n, y_{0,0}, ..., y_{0,n}))$$
$$...$$
$$\alpha_s(p) := a_s(med(p_{s+1}, ..., p_n, y_{s,0}, ..., y_{s,n-s}))$$
$$...$$
$$\alpha_{n-1}(p) := a_{n-1}(med(p_n, y_{n-1,0}, y_{n-1,1})).$$

and $\alpha_n(p) = x_{n,0}$. Similarly, let

$$\beta_0(p) := b_0(med(p_1, ..., p_n, x_{0,0}, ..., x_{n,0}))$$
$$...$$
$$\beta_t(p) = b_t(med(p_1, ..., p_{n-t}, x_{0,t}, x_{n-t,t}))$$
$$...$$
$$\beta_{n-1}(p) = b_{n-1}(med(p_1, x_{0,n-1}, x_{1,n-1}))$$

and $\beta_n(p) := y_{0,n}$. Finally, let

$$\varphi_1^{(a,b,x,y)}(R_N) = med(p_1, ..., p_n, \alpha_0(p), ..., \alpha_n(p))$$
$$\varphi_2^{(a,b,x,y)}(R_N) = med(p_1, ..., p_n, \beta_0(p), ..., \beta_n(p)).$$

Interestingly, the $(s+1)$-th highest cardinal function (with index s) for the first good does not depend on the s lowest peaks and the $(t+1)$-th lowest cardinal function (with index t) for the second good does not depend on the t highest peaks.

4.3.2 Double generalized median rules

A important subset of the class of *delta rules* is the one where each good is selected according to some fixed *generalized median rule* (Moulin, 1980). A rule is a *double generalized median rule* if its parameters (a, b, x, y) are such that there are two fixed vectors $(a_0^*, ..., a_n^*)$ and $(b_0^*, ..., b_n^*)$ such that

$$a_0^* \geq ... \geq a_n^*$$
$$b_0^* \leq ... \leq b_n^*$$

and for all s, t such that $s + t \leq n$, we have

$$a_s(y) = \min(a_s^*, y)$$
$$b_t(x) = \max(b_t^*, x)$$
$$x_{s,t} = a_s^*$$
$$y_{s,t} = b_t^*.$$

Among these rules, the *quantile rules* are those such that $(a_0^*, ..., a_n^*) \in \{0,1\}^{n+1}$ and $(b_0^*, ..., b_n^*) \in \{0,1\}^{n+1}$. They select each good according to a fixed quantile. An important quantile rule is the *extreme peaks rule*, which selects the lowest and highest peaks in the profile and corresponds to the parameters $(a_0^*, ..., a_n^*) = (0, ..., 0)$ and $(b_0^*, ..., b_n^*) = (1, ..., 1)$.

We will discuss the properties of these subclasses in Section 5.

4.3.3 Main characterization

We are now ready to present our main result, which says that the *delta rules* are characterized by *anonymity, strategy-proofness* and *continuity*.

Theorem 2 *Let φ be an n-person collective decision rule. Then, φ satisfies* anonymity, strategy-proofness *and* continuity *if and only if for some $(a, b, x, y) \in \mathcal{F}_n$,*

$$\varphi = \varphi^{(a,b,x,y)}.$$

It is clear from their definition that the *delta rules* satisfy *anonymity* and *continuity*. In the next lemma, we verify that these rules also satisfy *strategy-proofness*. We then show conversely that any rule satisfying these three properties must be a *delta rule*.

Lemma 2 *For all $(a, b, x, y) \in \mathcal{F}_n$, the rule $\varphi^{(a,b,x,y)}$ satisfies* strategy-proofness.

Proof. Let $i \in N$. Without loss of generality, we will assume that $i = n$, so that $N \setminus \{n\} = \{1, ..., n-1\}$. Let $R_{N\setminus\{n\}}$ be a fixed profile and let $p_1 \leq ... \leq p_{n-1}$ be the peaks in $p\left(R_{N\setminus\{n\}}\right)$ ranked in nondecreasing order. We will show that

$$\gamma : R_n \longrightarrow \left(\varphi_1^{(a,b,x,y)}(R_n, R_{N\setminus\{n\}}), \varphi_2^{(a,b,x,y)}(R_n, R_{N\setminus\{n\}})\right).$$

is a one agent *strategy-proof* rule, i.e. is of the form $\varphi^{(a_\gamma,b_\gamma)}$ for some $(a_\gamma,b_\gamma)\in\mathcal{F}$. Using the conventions $p_0:=0$ and $p_{n+1}:=1$, let j be the unique index in $\{0,...,n\}$ such that for all R_n, $\gamma_1(R_n)=p(R_n)$ only if $p(R_n)\in[p_j,p_{j+1}]$, . Similarly, let k be the unique index in $\{1,...,n-1\}$ such that for all R_n, $\gamma_2(R_n)=p(R_n)$ only if $p(R_n)\in[p_k,p_{k+1}]$. Then the formulas defining γ simplify as follows

$$\gamma_1(R_n)=med\begin{pmatrix}\max\{p_j,\alpha_{j+1}(\max\{p_{j+1},p(R_n)\},...,p_{n-1})\},p(R_n),\\ \min\{p_{j+1},\alpha_j(\max\{p(R_n),p_j\},p_{j+1},...,p_{n-1})\}\end{pmatrix}$$

and

$$\gamma_2(R_n)=med\begin{pmatrix}\max\left\{p_k,\beta_{n-(k+1)}(p_1,...,p_k,\min\{p_{k+1},p(R_n)\})\right\},\\ p(R_n),\min\{p_{k+1},\beta_{n-k}(p_1,...,\min\{p_k,p(R_n)\})\}\end{pmatrix}$$

which further simplify as follows

$$\begin{aligned}\gamma_1(R_n)&=med\begin{bmatrix}\max\{p_j,x_{j+1,k}\},p(R_n),\\ \min(p_{j+1},a_j(med[\max\{p_k,y_{j,k}\},p(R_n),\min\{p_{k+1},y_{j,k+1}\}]))\end{bmatrix}\\ \gamma_2(R_n)&=med\begin{bmatrix}\max\{p_k,b_{k+1}(\max\{p_j,x_{j+1,k}\},p(R_n),\min\{p_{j+1},x_{j,k}\})\},\\ p(R_n),\min\{p_{k+1},y_{j,k+1}\}\end{bmatrix}.\end{aligned}$$

The rule γ is then the rule $\varphi^{(a_\gamma,b_\gamma)}$ with parameters $b_\gamma:[\underline{a}_\gamma,\overline{a}_\gamma]\longmapsto[\underline{b}_\gamma,\overline{b}_\gamma]$ and $a_\gamma:[\underline{b}_\gamma,\overline{b}_\gamma]\longmapsto[\underline{a}_\gamma,\overline{a}_\gamma]$ such that

$$\begin{aligned}\underline{a}_\gamma&=\max\{p_j,x_{j+1,k}\}\\ \overline{a}_\gamma&=\min\{p_{j+1},x_{j,k}\}\\ \underline{b}_\gamma&=\max\{p_k,y_{j,k}\}\\ \overline{b}_\gamma&=\min\{p_{k+1},y_{j,k+1}\}\\ a^\gamma(y)&=\min\{p_{j+1},a_j(y)\}\\ b^\gamma(x)&=\max\{p_k,b_{k+1}(x)\}.\end{aligned}$$

Conditions (A), (B), (C) and (D) ensure that $(a_\gamma,b_\gamma)\in\mathcal{F}$. Therefore γ and $\varphi^{(a,b,x,y)}$ satisfy *strategy-proofness*, the desired conclusion.■

4.3.4 Proof of the "only if" implication in Theorem 2.

Throughout the proof, let $\varphi : \mathcal{R}^N \to [0,1]^{\{1,2\}}$ be an N-decision rule satisfying *strategy-proofness*, *continuity*, and *anonymity*. We will show that there is a list $(a,b,x,y) \in \mathcal{F}$ such that $\varphi = \varphi^{(a,b,x,y)}$.

Step 1: Preliminaries We first establish two useful Lemmas. The next Lemma says that if a rule satisfies *strategy-proofness*, a subset of agents with identical preferences cannot strictly gain by jointly misreporting an identical false preference.

Lemma 3 *Let $\varphi : \mathcal{R}^N \to [0,1]^{\{1,2\}}$ be strategy-proof. Then for all $S \subseteq N$, for all deviations $R, R' \in \mathcal{R}^N$, such that $R_i = R_j$ and $R'_i = R'_j$ for all $i, j \in S$, we have for all $i \in S$, $\varphi(R)\ R_i\ \varphi\left(R'_S, R_{N\setminus S}\right)$.*

Proof. Let $x, y \in [0,1]$. Without loss of generality, suppose that $S = \{1,...,s\}$. By *strategy-proofness*, for each $i \in \{0,...,s-1\}$, we have

$$\varphi\left(R'_{\{1,\ldots,i\}}, R_{\{i+1,\ldots,n\}}\right)\ R_i\ \varphi\left(R'_{\{1,\ldots,i+1\}}, R_{\{i,\ldots,n\}}\right).$$

Since $R_1, ..., R_s$ denote the same preference, it follows that $\varphi(R)\ R_1\ \varphi\left(R'_S, R_{N\setminus S}\right)$, the desired conclusion.■

Next, we observe that a rule that satisfies *strategy-proofness* and *continuity* must satisfy the following properties. The first property says that if some agents' peaks are initially to the left of the left good in the alternative selected by the rule at the initial profile, and their preferences change in such a way that their peak remains weakly to the left of the left good in the initially selected alternative, the rule selects the same alternative after and before the change.[6]

Left-uncompromisingness: For all $R, R' \in \mathcal{R}^N$, and $M \subseteq N$ such that for all $i \in M$, $p(R_i) < \varphi_1(R)$ and $p(R'_i) \leq \varphi_1(R)$, we have $\varphi\left(R'_M, R_{N\setminus M}\right) = \varphi(R)$.

[6] Ching (1997) introduces a similar property in the one good problem. Heo (2011) defines a similar property for the max extension domain. Both authors show that *uncompromisingness* is an implication of *strategy-proofness* and other properties.

We also define the symmetric property and other similar properties.

Right-uncompromisingness: For all $R, R' \in \mathcal{R}^N$, and $M \subseteq N$ such that for all $i \in M$, $\varphi_2(R) < p(R_i)$ and $\varphi_2(R) \leq p(R'_i)$, we have $\varphi\left(R'_M, R_{N\backslash M}\right) = \varphi(R)$.

Center-uncompromisingness: For all $R, R' \in \mathcal{R}^N$, and $M \subseteq N$ such that for all $i \in M$, $\varphi_1(R) < p(R_i) < \varphi_2(R)$ and $\varphi_1(R) \leq p(R'_i) \leq \varphi_2(R)$, we have $\varphi\left(R'_M, R_{N\backslash M}\right) = \varphi(R)$.

Left-trans-uncompromisingness: For all $R, R' \in \mathcal{R}^N$, and $M \subseteq N$ such that for all $i \in M$, $p(R_i) < \varphi_1(R)$ and $p(R'_i) = \varphi_2(R)$, we have $\varphi_2\left(R'_M, R_{N\backslash M}\right) = \varphi_2(R)$.

Right-trans-uncompromisingness: For all $R, R' \in \mathcal{R}^N$, and $M \subseteq N$ such that for all $i \in M$, $\varphi_2(R) < p(R_i)$ and $p(R'_i) = \varphi_1(R)$, we have $\varphi_1\left(R'_M, R_{N\backslash M}\right) = \varphi_1(R)$.

Lemma 4 *Let φ be a rule that satisfies strategy-proofness and continuity. Then φ satisfies peaks-selection, and the five forms of uncompromisingness listed above.*

Proof. We know from the results of Section 4.2 that all one-agent rules that satisfy *strategy-proofness* and *continuity* satisfy these properties. The n-agent rules inherit these properties, since the profile transformations that are consider can be achieved in $|M|$ steps by replacing the preference of one agent at a time.■

Step 1: We calibrate the rule φ and obtain a list (a, b, x, y) from φ. We first calibrate the rule. For each $M \subseteq N$, let $R^x_M \in \mathcal{R}^M$ be a preference profile such that $p(R^x_i) = x$ for all $i \in M$. From φ we derive $2n+2$ one-person decision rules as follows: for all (s,t) such that $s + t \leq n - 1$, let $\varphi^{s,t} : \mathcal{R} \to [0,1]^2$ be defined by

$$\varphi^{s,t}(R^x_n) = \varphi(R^0_{\{1,\ldots,s\}}, R^1_{\{s+1,\ldots,s+t\}}, R^x_{\{s+t+1,\ldots,n\}}) \text{ for all } x \in [0,1].$$

This function describes the locus of the allocation, as $n - (s+t)$ agents peaks move together from 0 to 1, while s agents peaks stay at 0 and t agents, who have already travelled, wait for the newcomers at 1.

Now by Lemma 3, for any pair (s,t) such that $s+t \leq n-1$, the rule $\varphi^{s,t}$ is a one-agent *strategy-proof* rule. Thus by Theorem 1, for any such pair, there exists $(a_{s,t}, b_{s,t}) \in \mathcal{F}$ such that $\varphi^{s,t} = \varphi^{(a_{s,t}, b_{s,t})}$.

For all (s,t) such that $s+t=n$, let

$$(x_{s,t}, y_{s,t}) := \varphi(R^0_{\{1,\ldots,s\}}, R^1_{\{s+1,\ldots,s+t\}}).$$

These definitions imply that for any such pair, $x_{s,t}$ is the lower end of the domain of $b_{s',t}(\cdot)$ for all $s' < s$ and that $y_{s,t}$ is the upper end of the domain of $a_{s,t'}(\cdot)$ for all $t' < t$. For any pair (s,t) such that $s+t \leq n-1$, let $x_{s,t}$ be the upper end of the domain of $b_{s,t}(\cdot)$ and let $y_{s,t}$ be the lower end of the domain of $b_{s,t}(\cdot)$. Thus, for all $s+t \leq n-1$, we obtain

$$a_{s,t} : [y_{s,t}, y_{s,n-s}] \longrightarrow [x_{n-t,t}, x_{s,t}]$$
$$b_{s,t} : [x_{n-t,t}, x_{s,t}] \longrightarrow [y_{s,t}, y_{s,n-s}].$$

Last, for all $s \in \{0,...,n-1\}$, let

$$a_s := a_{s,0}$$

which maps $[y_{s,0}, y_{s,n-s}]$ to $[x_{n,0}, x_{s,0}]$ and for all $t \in \{0,...,n\}$, let

$$b_t := b_{0,t}$$

which maps $[x_{n-t,t}, x_{0,t}]$ to $[y_{0,t}, y_{0,n}]$.

Step 2: We verify that (a,b,x,y) is indeed an element of $\mathcal{F}$.

Lemma 5 *For all (s,t) such that $s+t \leq n-1$, the inequalities $(A1)$, $(A2)$ and $(A3)$ hold.*

Proof. Let (s,t) be such that $s+t \leq n-1$. Let $i_1,...,i_{s+1}$ and $j_1,...,j_t$ be distinct agents in N, and let $M_0 := \{i_1,...,i_{s+1}\}$, $M_1 := \{j_1,...,j_t\}$ and $M_2 := N \setminus (M_0 \cup M_1)$.

First, we show that $x_{s+1,t} \leq x_{s,t}$. Let $x := x_{s+1,t}$. If $x_{s+1,t} = 0$, the inequality is true. Suppose that $x_{s+1,t} > 0$. Let R be a profile such that $R_{M_0} := R^0_{M_0}$, $R_{M_1} := R^1_{M_1}$ and if $M_2 \neq \emptyset$, $R_{M_2} := R^x_{M_2}$. By definition of $x_{s+1,t}$, we have $\varphi_1(R) = x_{s+1,t}$.

Since $x_{s+1,t} > 0$, by *left-uncompromisingness*, we have $\varphi_1(R^x_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}) = x_{s+1,t}$. Therefore $\varphi_1^{s,t}(R^x_n) = x_{s+1,t}$. By definition of $x_{s,t}$, this implies that $x_{s+1,t} \leq x_{s,t}$, the desired inequality. By a symmetric argument, we also obtain that $y_{s,t} \leq y_{s,t+1}$.

Second, we show that $y_{s,t} \leq y_{s+1,t}$. Let $y := y_{s+1,t}$. Let R be the profile such that $R_{M_0} := R^0_{M_0}$, $R_{M_1} := R^1_{M_1}$ and if $M_2 \neq \emptyset$, $R_{M_2} := R^y_{M_2}$. By definition of $y_{s+1,t}$, we have $\varphi_2(R) = y_{s+1,t}$. By *left-trans-uncompromisingness*, we have $\varphi_2(R^y_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}) = y_{s+1,t}$. Therefore $\varphi_y^{s,t}(R^y_n) = y_{s+1,t}$. By definition of $y_{s,t}$, this implies that $y_{s,t} \leq y_{s+1,t}$, the desired inequality. By a symmetric argument, we also obtain that $x_{s,t+1} \leq x_{s,t}$.

Last, we show that $x_{s+1,t} \leq x_{s,t+1}$. Let $x := x_{s+1,t}$. Let R be a profile such that $R_{M_0} := R^0_{M_0}$, $R_{M_1} := R^1_{M_1}$ and if $M_2 \neq \emptyset$, $R_{M_2} := R^x_{M_2}$. By definition of $x_{s+1,t}$, we have $\varphi_1(R) = x_{s+1,t}$. By *strategy-proofness*, $\varphi(R)\ R^0_{i_{s+1}}\ \varphi\left(R^1_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}\right)$, therefore $\varphi_1(R) \leq \varphi_1\left(R^1_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}\right)$. If $M_2 = \emptyset$, then by definition of $x_{s,t+1}$, we have $\varphi_1\left(R^1_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}\right) = x_{s,t+1}$. Otherwise $\varphi_1\left(R^1_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}\right) = \varphi_1^{s,t+1}(R^x_n)$ and by definition of $x_{s,t+1}$, we have $\varphi_1^{s,t+1}(R^x_n) \leq x_{s,t+1}$. In both cases

$$x_{s+1,t} = \varphi_1^{s+1,t}(R^x_n) = \varphi_1(R) \leq \varphi_1\left(R^1_{i_{s+1}}, R_{N\backslash\{i_{s+1}\}}\right) \leq x_{s,t+1},$$

the desired inequality. By a symmetric argument, we also obtain that $y_{s+1,t} \leq y_{s,t+1}$.■

Lemma 6 *For all* $y \in [y_{s,t}, y_{s,n-t}]$, $a_{s,t}(y) = a_s(y)$ *and for all* $x \in [x_{n-t,t}, x_{s,t}]$, $b_{s,t}(x) = b_t(x)$.

Proof. The first equality follows immediately from *left-uncompromisingness*, the second from *right-uncompromisingness*.■

Lemma 7 *For all* (s,t) *such that* $s + t \leq n - 1$, *the inclusions*

$$a_s([y_{s,t}, y_{s,t+1}]) \subseteq [\max\{x_{s+1,t}, y_{s,t}\}, \max\{x_{s,t}, y_{s,t+1}\}]; \tag{B1}$$

$$b_t([x_{s+1,t}, x_{s,t}]) \subseteq [\max\{x_{s+1,t}, y_{s,t},\}, \max\{x_{s,t}, y_{s,t+1}\}]; \tag{B2}$$

hold.

Proof. We first prove the second inclusion. Since the restriction of a_s to $[y_{s,t}, y_{s,n-t}]$ and the restriction of b_t to $[x_{n-t,t}, x_{s,t}]$ form a pair in $\mathcal{F}$, therefore b_t maps $[x_{n-t,t}, x_{s,t}]$ to $[y_{s,t}, y_{s,n-t}]$. Moreover, for all $x \geq x_{s+1,t}$ we have $b_t(x) \geq x \geq x_{s+1,t}$ It follows that

$$b_t([x_{s+1,t}, x_{s,t}]) \subseteq [\max\{x_{s+1,t}, y_{s,t},\}, y_{0,n}].$$

We still need to show that for all $x \in [x_{s+1,t}, x_{s,t}]$, $b_t(x) \leq \max\{x_{s,t}, y_{s,t+1}\}$. Let $\psi : \mathcal{R} \rightarrow [0,1]^2$ be the one agent rule defined by

$$\psi(R_n^x) = \varphi(R^0_{\{1,...,s\}}, R^1_{\{s+1,...,s+t\}}, R^{x_{s,t}}_{\{s+t+1,...,n-1\}}, R_n^x).$$

By Theorem 1 there is a pair $(a_\psi, b_\psi) \in \mathcal{F}$ such that $\psi = \varphi^{(a_\psi, b_\psi)}$. Let $[\underline{a}_\psi, \overline{a}_\psi]$ and $[\underline{b}_\psi, \overline{b}_\psi] \subseteq [0,1]$ be such that

$$b_\psi : [\underline{a}_\psi, \overline{a}_\psi] \longmapsto [\underline{b}_\psi, \overline{b}_\psi]$$
$$a_\psi : [\underline{b}_\psi, \overline{b}_\psi] \longmapsto [\underline{a}_\psi, \overline{a}_\psi].$$

One shows that $[\underline{a}_\psi, \overline{a}_\psi] = [x_{s+1,t}, x_{s,t}]$ and $[\underline{b}_\psi, \overline{b}_\psi] = [\max\{x_{s,t}, y_{s,t}\}, \max\{x_{s,t}, y_{s,t+1}\}]$ and for all $x \in [x_{s+1,t}, x_{s,t}]$, $b_\psi(x) = b_t(x)$. Therefore $b_t(x) \leq \max\{x_{s,t}, y_{s,t+1}\}$, which proves the first inclusion. The second inclusion follows from a symmetric argument.■

Lemma 8 *The list* (a, b, x, y) *satisfies conditions* $(B3)$, $(B4)$, $(B5)$ *and* (C).

Proof. From 6, we know that the parameters of $\varphi^{s,t}$ are

$$a_s : [y_{s,t}, y_{s,n-s}] \longrightarrow [x_{n-t,t}, x_{s,t}]$$
$$b_t : [x_{n-t,t}, x_{s,t}] \longrightarrow [y_{s,t}, y_{s,n-s}].$$

and that $(a_s, b_t) \in \mathcal{F}$. The four conditions follow easily from this observation.■

Lemma 9 *The list* (a, b, x, y) *satisfies condition* (D).

Proof. Let $y \in [y_{s+1,0}, y_{s+1,n-(s+1)}]$. Suppose by contradiction that $a_s(y) < a_{s+1}(y)$. At the profile

$$(R_1^0, ..., R_{s+1}^0, R_{s+2}^y, ..., R_n^y),$$

the agent $s+1$ can change the outcome from $(a_{s+1}(y), y)$ to $(a_s(y), y)$ and strictly benefit from it, which contradicts *strategy-proofness*. The second inequality follows from a symmetric argument.■

Lemma 10 *The list* $(a, b\, x, y)$ *satisfies condition* (E).

Proof. We first prove the first inequality. Let $x \in [x_{s,n-s}, x_{s,0}] \cap b_{n-s}^{-1}([y_{s,0}, y_{s,n-s}])$. Let $y \in [y_{s,0}, y_{s,n-s}]$ be such that $y = b_{n-s}(x)$. Suppose by contradiction that $x < a_s(b_{n-s}(x))$. Then

$$\varphi\left(R_1^x, ..., R_s^x, R_{s+1}^1, ..., R_n^1\right) = (x, b_{n-s}(x)).$$

By *strategy-proofness*, this implies that

$$\varphi\left(R_1^x, ..., R_s^x, R_{s+1}^y, ..., R_n^y\right) = (x, b_{n-s}(x)). \tag{2}$$

We also have

$$\varphi\left(R_1^0, ..., R_s^0, R_{s+1}^y, ..., R_n^y\right) = (a_s(y), y).$$

By *strategy-proofness* and because $x < a_s(b_{n-s}(x))$, this implies that

$$\varphi\left(R_1^x, ..., R_s^x, R_{s+1}^y, ..., R_n^y\right) = (a_s(y), y)$$

which contradicts equation (2) and proves the first inequality. The second inequality follows from a symmetric argument.■

Step 3: We now show that $\varphi = \varphi^{(a,b,x,y)}$, where $(a, b, x, y) \in \mathcal{F}$ is obtained from φ in Step 1. In the case where $n = 1$, the equality follows from Theorem 1. For $n \geq 2$, we distinguish two cases, depending on the number of agents.

Case $n = 2$. Let $(R_1, R_2) \in \mathcal{R}^{\{1,2\}}$, we will show that $\varphi(R_1, R_2) = \varphi^{(a,b,x,y)}(R_1, R_2)$. Without loss of generality, suppose that $p(R_1) \leq p(R_2)$. We have

$$\varphi\left(R^0, R_2\right) = \varphi^{1,0}(R_2) = \varphi^{(a,b,x,y)}\left(R^0, R_2\right).$$

By *left-uncompromisingness*, if $p(R_1) \leq \varphi_1^{0,1}(R_2)$, we have

$$\varphi(R_1, R_2) = \varphi^{1,0}(R_2) = \varphi^{(a,b,x,y)}(R_1, R_2).$$

By a symmetric argument, we obtain that if $p(R_2) \geq \varphi_2^{0,1}(R_1)$, we have

$$\varphi(R_1, R_2) = \varphi^{0,1}(R_1) = \varphi^{(a,b,x,y)}(R_1, R_2).$$

Next, if $p(R_1) = p(R_2) \in [x_{0,0}, y_{0,1}]$, we have

$$\varphi(R_1, R_2) = \varphi^{0,0}(R_2) = \varphi^{(a,b,x,y)}(R_1, R_2).$$

Thus, if $p(R_2) \in [x_{0,0}, y_{0,1}]$ and $\varphi_1^{0,0}(R_2) \leq p(R_1) \leq p(R_2)$, we have

$$\varphi(R_1, R_2) = \varphi^{0,0}(R_2) = \varphi^{(a,b,x,y)}(R_1, R_2).$$

By an analogous reasoning, we obtain that if $p(R_1) \in [x_{1,0}, y_{0,0}]$ and $p(R_1) \leq p(R_2) \leq \varphi_2^{0,0}(R_1)$, we have

$$\varphi(R_1, R_2) = \varphi^{0,0}(R_2) = \varphi^{(a,b,x,y)}(R_1, R_2).$$

Last if $\varphi_1^{1,0}(R_2) \leq p(R_1) \leq \varphi_1^{0,0}(R_2)$ and $\varphi_2^{0,0}(R_1) \leq p(R_2) \leq \varphi_2^{0,1}(R_1)$, Theorem 1 implies that

$$\varphi(R_1, R_2) = (p(R_1), p(R_2)) = \varphi^{(a,b,x,y)}(R_1, R_2),$$

which shows the equality for any profile (R_1, R_2) such that $p(R_1) \leq p(R_2)$, the desired conclusion.

Case $n \geq 3$. We show that for all R, we have $\varphi(R) = \varphi^{(a,b,x,y)}(R)$. The proof is by induction on the number n_R of interior peaks in the profile R, i.e.

$$n_R := |\{x \in p(R) : 0 < x < 1\}|.$$

If $n_R = 1$, let $s := |\{i \in N : p(R_i) = 0\}|$ and let $t := |\{i \in N : p(R_i) = 1\}|$. Then

$$\varphi(R) = \varphi^{s,t}(R) = \varphi^{(a,b,x,y)}(R)$$

where the first equality holds by anonymity and the definition of $\varphi^{s,t}$ and the second equality holds by the definition of $\varphi^{(a,b,x,y)}$.

If $n_R = 2$, the equality follows from Case $n = 2$.

Let $n' \geq 1$ and suppose that the claim is true for all $R \in \mathcal{R}^N$ such that $n_R = n'$. We will show that it is then true for all R such that $n_R = n' + 1$. Let R be an arbitrary profile such that $n_R = n' + 1$. Let $i, j \in N$ such that $0 < p(R_i) < 1$ and $0 < p(R_j) < 1$. Holding $R_{N \setminus \{i,j\}}$ constant, consider the rules

$$\varphi_* : (R'_i, R'_j) \to \varphi(R'_i, R'_j, R_{N \setminus \{i,j\}})$$

and

$$\varphi'_* : (R'_i, R'_j) \to \varphi^{(a,b,x,y)}(R'_i, R'_j, R_{N \setminus \{i,j\}}).$$

The rule φ_* inherits *strategy-proofness* from φ. The rule φ'_* inherits *strategy-proofness* from $\varphi^{(a,b,x,y)}$, which is *strategy-proof* by Lemma 2. Consider the parameters obtained from calibrating both of these rules. Since the claim is true for all R such that $n_R = n'$, the parameters of these two rules are the same. By the Case $n = 2$, it follows that these two rules are equal. Since this is true for all profile R such that $n_R = n' + 1$, we conclude that the claim is true for any such profile. Thus the claim is true for all profile $R \in \mathcal{R}^N$, the desired conclusion.■

5 Subclasses

In this section we examine subclasses of rules of interest. In particular, we characterize the *delta rules* that satisfy additional properties.

5.1 Group strategy-proofness

Group strategy-proofness means that no coalition of agents can jointly misreport their preferences in such a way that each member of the coalition weakly gains with at least one strict gain.

Group strategy-proofness: For all $S \subseteq N$, all S-deviations $R, R' \in \mathcal{R}^N$, if for some $i \in S$, $\varphi(R')\ P_i\ \varphi(R)$, then for some $t \in S$, $\varphi(R)\ P_t\ \varphi(R')$.

One can easily verify that all the rules characterized in the previous section also satisfy *group strategy-proofness.*

Theorem 3 *For all $(a, b, x, y) \in \mathcal{F}$, the rule $\varphi = \varphi^{(a,b,x,y)}$ satisfies group strategy-proofness.*

Remark 2 *There is no logical relation between group strategy-proofness in the lexmax and the max extension. However, within the rules that are strategy-proof in the lexmax extension, one can show that the set of rules that are group strategy-proof in the lexmax extension contains the set of rules that are group-strategy-proof in the max extension. This is because, except in the case of a coalition of agents with the same preferences, there are more group improvements (which are strict gains for at least one agent) under the max extension than under the lexmax extension. Theorem 3 contrasts with the result in Heo (2011) which shows that only a subset of the double generalized median rules (with fixed cardinal points) are group-strategy-proof.*

5.2 Unanimity

By *unanimity*, for every preference profile such that all agents have the same peak, both goods must be located at the agents common peak.

Unanimity: For all $R \in \mathcal{R}^N$ such that $p(R_1) = ... = p(R_n)$, we have $\varphi(R_N) = (p(R_1), p(R_1))$.

Theorem 4 *For all $(a, b, x, y) \in \mathcal{F}$, the rule $\varphi = \varphi^{(a,b,x,y)}$ satisfies unanimity if and only if $x_{0,0} = 1$, $y_{0,0} = 0$, and $a_0(x) = b_0(x) = x$ for all $x \in [0, 1]$.*

5.3 Pareto-optimality

By *Pareto-optimality*, for every preference profile the assigned alternative cannot be changed such that no agent is worse off and some agent is better off.

Pareto-optimality: For all $R \in \mathcal{R}^N$ and all $x \in [0, 1]^2$, if for some $i \in N$, $x \; P_i \; \varphi(R)$, then for some $t \in N$, $\varphi(R) \; P_t \; x$.

Notice that for *Pareto-optimality* it is not sufficient that the assigned alternative is an element of $[\underline{p}(R), \overline{p}(R)]^2$. It is necessary for every assigned alternative that each closed interval with endpoints of two assigned locations contains at least one reported peak of the agents.

Lemma 11 *Let φ be a decision rule. Then, φ satisfies Pareto-optimality if and only if for all $R \in \mathcal{R}^N$, $\varphi(R) \in [\underline{p}(R), \overline{p}(R)]^2$ and there exists $i \in N$ such that $p(R_i) \in [\varphi_1(R), \varphi_2(R)]$.*

We now characterize the subclass of rules that satisfy *Pareto-optimality*.

Theorem 5 *For all $(a, b, x, y) \in \mathcal{F}$, the rule $\varphi = \varphi^{(a,b,x,y)}$ satisfies Pareto-optimality if and only if*

1. *For all $r \in \{0, \ldots, n\}$, $y_{r,0} = 0$ and $x_{0,r} = 1$;*
2. *For all $r \in \{1, \ldots, n-1\}$, either $x_{r,n-r} = 0$ or $y_{r,n-r} = 1$.*

The first condition says that (i) when the set of the preferred locations of the agents is $\{0, x\}$, for some $x \in [0,1]$, the rule locates both goods in $[0, x]$; (ii) when the set of the preferred locations of the agents is $\{x, 1\}$, for some $x \in [0,1]$, the rule locates both goods in $[x, 1]$; (iii) when all agents have the same preferred location, the rule locates both goods at this location. The second condition says that if all agents are extremists, i.e. their preferred location is either 0 or 1, then at least one of the two locations is an extreme, either 0 or 1. Therefore, Conditions 1. and 2. are necessary for Pareto-optimality.

In fact, these conditions are also sufficient. The first condition ensures that both locations are contained in the closed interval defined by the lowest and highest peak in the profile. The second condition ensures that the rule never locates both goods in the open interval defined by two consecutive peaks. These are precisely the conditions that characterize Pareto-optimality in this model. Therefore, Conditions 1. and 2. are sufficient for Pareto-optimality.

Remark 3 *Except at unanimous profiles, Pareto-optimality with respect to the max extension implies Pareto-optimality with respect to the lexmax extension.*

5.4 Generalized medians

In the max extension domain, Heo (2011) shows that the double generalized median rules (with fixed cardinal points) are the only ones that satisfy *strategy-proofness*, *anonymity*, *continuity* and *users-only*, and property that requires that the location of a good only depends on the preferences of the agents who prefer it in the menu.

In the max extension domain, Miyagawa (1998) provides various characterizations of the extreme rule. In particular, he shows that it is the only one that satisfies on the one hand, *strategy-proofness*, *continuity* and *Pareto-efficiency*, and on the other hand *group strategy-proofness*, *peaks-selection* and *citizen sovereignty*.[7] We provide a new axiomatization of this rule in the next subsection.

[7] By *citizen sovereignty*, all alternatives are attainable via some reported profile. *Peaks-selection* is defined in the next section.

5.5 Diversity

By *diversity*, the two goods cannot coincide, unless all agents have the same preferred location.

Diversity: For all $R \in \mathcal{R}^N$,

$$|\{p(R_i) : i \in N\}| > 1 \Rightarrow \varphi_1(R) \neq \varphi_2(R).$$

We have the following results.

Theorem 6 *For all $(a, b, x, y) \in \mathcal{F}$, the rule $\varphi = \varphi^{(a,b,x,y)}$ satisfies diversity if and only if $x_{0,1} < y_{0,1}$ and $x_{1,0} < y_{1,0}$.*

Moreover, the only rule within this class that satisfies *Pareto-optimality* is the extreme peaks rule.

Theorem 7 *For all $(a, b, x, y) \in \mathcal{F}$, the rule $\varphi = \varphi^{(a,b,x,y)}$ satisfies Pareto-optimality and diversity and if and only if it is the extreme peaks rule.*

5.6 Population-monotonicity

In a model with a variable population, Ehlers (2003) characterizes a class of rules, which he calls the single-plateau rules. In this context, a rule satisfies *population-monotonicity* (Thomson, 1983a, 1983b) if when new agents are added to the population and given preferences, and the preferences of the agents initially present are kept fixed, these agents either all weakly gain or they all weakly lose. Ehlers (2003) shows that in his model, a rule satisfies *Pareto-optimality* and population-monotonicity if and only if it is a single-plateau rule.[8]

Next, we describe the class of single-plateau rules, and show that it is a subclass of the *Pareto-optimal delta rules*, characterized in Section 5.3. A rule $\varphi^{(a,b,x,y)}$ is

[8] Miyagawa (1998) characterizes the set of rules that satisfy *Pareto-optimality* and *population-monotonicity* in the max extension model. This family contains the left peaks-rule, the right-peaks rule, the extreme peaks rule and complicated combinations of these three rules. The only rule in this family that satisfies *continuity* is the extreme peaks rule.

single-plateau, if

$$x_{n,0} = y_{n,0} = 0$$
$$x_{0,n} = y_{0,n} = 1$$
$$y_{s,0} = 0 \text{ for all } s = 1, ..., n-1$$

and there are two pairs (x^*, y^*) and (x°, y°) in $[0,1]^2$ such that $x^* \leq y^*$, $x^\circ \leq x^*$ and $y^* \leq y^\circ$ and either $x^\circ = 0$ or $y^\circ = 1$ and

$$x_{s,t} = x^* \text{ for all } s,t \text{ such that } 0 < s+t < n$$
$$y_{s,t} = y^* \text{ for all } s,t \text{ such that } s+t < n \text{ and } t > 0$$
$$x_{s,n-s} = x^\circ \text{ for all } s = 1, ..., n-1$$
$$y_{s,n-s} = y^\circ \text{ for all } s = 1, ..., n-1$$
$$b_t(x) = f(x) \text{ for all } t = 2, ..., n-2.$$

and a continuous and strictly decreasing function $f : [x^\circ, x^*] \to [y^*, y^\circ]$ such that

$$a_0(y) = y$$
$$a_1(y) = a_{n-1}(y) = \min\left\{y, y^*, f^{-1}(y)\right\}$$
$$a_s(y) = f^{-1}(y) \text{ for all } s = 2, ..., n-2$$
$$a_n(y) = 0$$

and

$$b_0(x) = x$$
$$b_1(x) = b_{n-1}(x) = \max\{f(x), y^*, x\}$$
$$b_t(x) = f(x) \text{ for all } t = 2, ..., n-2.$$
$$b_n(x) = 1$$

Gordon (2007) has shown that in any pure public good model (fixed set of alternatives that does not depend on the set of agents; symmetric preference domain) with a variable population, under *Pareto-optimality*, *population-monotonicity* is equivalent to *strategy-proofness* and *represented-types-only*, a property that requires that the rule only depends on the set (and not the profile) of preferences $\{R_i : i \in N\}$. Ehlers' (2003) characterization and ours illustrate this link. The single-plateau rules are precisely the *Pareto optimal delta rules*, characterized in Section 5.3, which satisfy *represented types only*.[9]

5.7 Replacement-domination

In a model with a fixed population, Ehlers (2002) characterizes a class of the rules, which he calls the single-peaked rules. In this context, a rule satisfies *replacement-domination* (Moulin, 1987) if when the preferences of one agent changes, while the other agent's preferences are kept fixed, these agents either all weakly gain or they all weakly lose. Ehlers (2002) shows that in his model, a rule satisfies *Pareto-optimality* and *replacement-domination* if and only if it is a single-peaked rule.[10]

The single-peaked rules form a subclass of the *Pareto-optimal delta rules*, that we characterized in section 5.3. A rule $\varphi^{(a,b,x,y)}$ is a single-peaked rule, if it is a single-plateau rule and in addition $x^* = y^*$.[11]

6 Peaks-selection and quantile rules

In this section, we examine the implications of requiring that each public should be provided on some reported peak.

We address this question in a more general model than the two goods setup considered in the rest of the paper. We assume here that m public goods, $m \in \mathbb{N}$, have to be selected. Let $M = \{1, \ldots, m\}$. An *alternative* is a tuple $x = (x_1, \ldots, x_m)$ such that $0 \leq x_1 \leq \cdots \leq x_m \leq 1$. A *decision rule* is a mapping φ associating to every profile $R \in \mathcal{R}^N$ an alternative. We write $\varphi(R) = (\varphi_1(R), \ldots, \varphi_m(R))$. Preferences are extended over the set of alternatives. Each agent compares two alternatives via the *lexmax ordering* induced by his single-peaked preference. Therefore, for two m-tuples

[9] It would be interesting to determine which among the rules in the class characterized in Section 4 satisfy *population-monotonicity*. The class of such rules strictly contains the single-plateau rules.

[10] Miyagawa (2001) shows that the only rules that satisfy *Pareto-optimality* and *replacement-domination* in the max extension model are the left peaks rule and the right-peaks rule. The left-peaks rule selects the two lowest peaks in the profile. The right-peaks rule selects the two highest.

[11] It would be interesting to determine which among the rules in the class characterized in Section 4 satisfy *replacement-domination*. The class of such rules strictly contains the class of single-peaked rules.

x, x' and two permutations τ, τ' of M such that $x_{\tau(1)}\ R_i\ x_{\tau(2)}\ R_i \ldots R_i\ x_{\tau(m)}$ and $x'_{\tau'(1)}\ R_i\ x'_{\tau'(2)}\ R_i \ldots R_i\ x'_{\tau'(m)}$ we have $(x_1, \ldots, x_m)\ P_i\ (x'_1, \ldots, x'_m)$ if there exists $t \in M$ such that for all $k < t$, $x_{\tau(k)}\ I_i\ x'_{\tau'(k)}$ and $x_{\tau(t)}\ P_i\ x'_{\tau'(t)}$. If for all $k \in M$, $x_{\tau(k)}\ I_i\ x'_{\tau'(k)}$, then $x\ I_i\ x'$.

By *peaks-selection*, each public good has to provided on some reported peak.

Peaks-selection: For all $R \in \mathcal{R}^N$, $\{\varphi_k(R) \mid k \in M\} \subseteq \{p(R_i) \mid i \in N\}$.

Note that (by some adapted version of Lemma 11) *peaks-selection* implies *Pareto-optimality*. In general, this implication does not hold in the max extension model studied by Miyagawa (1998) and Heo (2011).

It is easy to see that when $m = 2$, the only *delta rules* that satisfy peaks selection are the quantile rules. We will show here that this fact is a consequence of a much stronger result that holds more generally for any $m \geq 1$.

In the general case $m \geq 1$, a quantile rule can be described by m vectors, in the same way we introduced these rules in the case where $= 2$. A simpler way of describing a quantile rule is as follows. Denote by $\vec{m} = (m_k)_{k=1,\ldots,n} \in \{0, 1, \ldots, m\}^n$ an n-vector such that $\sum_{k=1}^{n} m_k = m$. For any profile $R \in \mathcal{R}^N$, let $p_1 \leq \ldots \leq p_n$ be the peaks in the profile ranked in nondecreasing order. The quantile rule $\varphi^{\vec{m}}$ is such that for any $R \in \mathcal{R}^N$, m_k goods are located on peak p_k, for each $k = 1, \ldots, n$. We have the following result.

Theorem 8 *Let $m \geq 2$. The quantile rules are the only decision rules satisfying anonymity, continuity and peaks-selection.*

Proof. It is easy to show that a quantile rule satisfies there properties. By *anonymity*, the rule φ can only depend on $p_1, \ldots, p_n$, in the sense that any two preference profiles such that the lists $p_1, \ldots, p_n$ are the same yield the same selection. Between any two peak profiles p and p' such that $p_1 < \ldots < p_n$ and $p'_1 < \ldots < p'_n$, there exists a continuous path of peak profiles satisfying the same strict inequalities that connects them. By continuity of φ along the path, it is clear that if φ coincides with a quantile rule at one profile on the path, it coincides with it on the entire path. Therefore there is some fixed quantile rule $\varphi^{\vec{m}}$ that φ coincides with on any profile p such that

$p_1 < ... < p_n$. Since this subset of profiles is dense, the two rules are the same on their entire domain, therefore φ is a quantile rule.■

7 Condorcet Winners

In this section we consider the existence of Condorcet winners. For a given profile of preferences, a Condorcet winner is an alternative which beats any other alternative by majority voting at this profile. Formally, for $R \in \mathcal{R}^N$, the set of Condorcet winners is given by

$$CW(R) = \{x \in [0,1]^2 \mid \text{ for all } y \in [0,1]^2, |\{i \in N \mid xR_iy\}| \geq |\{i \in N \mid yR_ix\}|\}. \quad (3)$$

Moulin (1980) showed that for one public good and odd n, the Condorcet winner is unique for every reported profile and it is the median of the announced peaks. Klaus and Storcken (1999) showed that for even n, the set of Condorcet winners is the closed interval with endpoints $p(R_{\frac{n}{2}})$ and $p(R_{\frac{n}{2}+1})$ when $p(R_1) \leq p(R_2) \leq \cdots \leq p(R_n)$. In this section we address the question whether Condorcet winners always exist, and if they exist, are they unique? The following lemma is an obvious consequence of (3).

Lemma 12 *Let $R \in \mathcal{R}^N$. For all permutations σ of N, $CW(R^\sigma) = CW(R)$. Moreover, if $x \in CW(R)$, then x is Pareto-optimal at R.*

By Lemma 12, for $R \in \mathcal{R}^N$ we may assume that $p(R_1) \leq p(R_2) \leq \cdots \leq p(R_n)$. For $n = 4$ we derive a positive result.

Theorem 9 *Let $n = 4$ and $R \in \mathcal{R}^N$. Then, $(p(R_2), p(R_3)) \in CW(R)$. Moreover, if $p(R_1) < p(R_2) < p(R_3) < p(R_4)$, then $CW(R) = \{(p(R_2), p(R_3))\}$.*

Proof. Let $R \in \mathcal{R}^N$. First, we show that $(p(R_2), p(R_3)) \in CW(R)$. Thus, we have to prove for all $y \in [0,1]^2$,

$$|\{i \in N \mid (p(R_2), p(R_3))R_iy\}| \geq |\{i \in N \mid yR_i(p(R_2), p(R_3))\}|. \quad (4)$$

We consider different cases.

Case 1: $\{y_1, y_2\} \cap \{p(R_2), p(R_3)\} = \emptyset$.

Then, $(p(R_2), p(R_3))P_2y$ and $(p(R_2), p(R_3))P_3y$. Since $n = 4$, (4) holds.

Case 2: $y_1 = p(R_2)$.

Thus, let $y_2 \neq p(R_3)$. If $y_2 < p(R_3)$, then $(p(R_2), p(R_3))P_3y$ and $(p(R_2), p(R_3))P_4y$. Since $n = 4$, (4) holds. If $y_2 > p(R_3)$, then $(p(R_2), p(R_3))P_1y$ and $(p(R_2), p(R_3))P_2y$. Since $n = 4$, (4) holds.

Case 3: $y_1 = p(R_3)$.

This case is analogous to Case 2.

We have shown $(p(R_2), p(R_3)) \in CW(R)$. To show the last assertion of the theorem, let $p(R_1) < p(R_2) < p(R_3) < p(R_4)$. We have to prove $CW(R) = \{(p(R_2), p(R_3))\}$. Thus, assume by contradiction, $y \in CW(R)$ and $y \neq (p(R_2), p(R_3))$. Let $y_1 \leq y_2$.

Case 1: $y_1 < p(R_2)$.

Then, $(p(R_2), y_2)P_2y$, $(p(R_2), y_2)P_3y$ and $(p(R_2), y_2)P_4y$. Hence, $y \notin CW(R)$.

Case 2: $y_2 > p(R_3)$. This case is analogous to Case 1.

Case 3: $p(R_2) \leq y_1$ and $y_2 \leq p(R_3)$.

By Lemma 12, y is *Pareto-optimal* for R. Thus, by Lemma 11 $p(R_2) = y_1$ or $y_2 = p(R_3)$. Without loss of generality, assume that $y_1 = p(R_2)$ and $y_2 < p(R_3)$. Then, $(p(R_1), p(R_3))P_1y$, $(p(R_1), p(R_3))P_3y$ and $(p(R_1), p(R_3))P_4y$. Thus, $y \notin CW(R)$. This completes the proof of Theorem 9. ■

Remark 4 *If* $p(R_1) = p(R_2)$, *then*

$$CW(R) = \{(p(R_1), \lambda p(R_1) + (1 - \lambda)p(R_3)) \mid \lambda \in [0, 1]\}.$$

For $n = 2$ it is easy to see that $CW(R)$ coincides with the *Pareto-optimal* alternatives at R. However, these are the only positive results. For different n the set of Condorcet winners may be empty.

Theorem 10 *Let* $n \geq 3$ *and* $n \neq 4$. *If* $R \in \mathcal{R}^N$ *such that* $|\{p(R_i) \mid i \in N\}| = n$, *then* $CW(R) = \emptyset$.

Proof. Let $R \in \mathcal{R}^N$ such that $p(R_1) < p(R_2) < \cdots < p(R_n)$. We have to show $CW(R) = \emptyset$. Suppose that $y \in CW(R)$. We consider two cases.

Case 1: n is odd. By $med(p(R))$ we denote the median of the reported peaks. If $y_1 < med(p(R))$, then by Moulin (1980) it follows that at least $\frac{n}{2}$ agents strictly prefer $(med(p(R)), y_2)$ to y which implies $y \notin CW(R)$. Analogously, it follows that if $y_2 > med(p(R))$, then $y \notin CW(R)$. Since $y_1 \leq y_2$, we must have $y = (med(p(R)), med(p(R)))$. Then, y is not *Pareto-optimal*, contradicting Lemma 12. Thus, $y \notin CW(R)$ and $CW(R) = \emptyset$.

Case 2: n is even. If $y_1 < p(R_{\frac{n}{2}})$, then $|\{i \in N \,|\, (p(R_{\frac{n}{2}}), y_2) P_i y\}| \geq \frac{n}{2} + 1$. Hence, $y \notin CW(R)$. Analogously it can be shown, that $y_2 > p(R_{\frac{n}{2}+1})$ implies $y \notin CW(R)$. Thus, $p(R_{\frac{n}{2}}) \leq y_1 \leq y_2 \leq p(R_{\frac{n}{2}+1})$. But then, since $n \geq 6$,

$$|\{i \in N \,|\, (p(R_{\frac{n}{2}-1}), p(R_{\frac{n}{2}+2})) P_i y\}| \geq n - 2 > \frac{n}{2}.$$

Hence, $y \notin CW(R)$ and $CW(R) = \emptyset$. ■

References

Barberà S., (2011) Strategy-proof Social Choice, in *Handbook of Social Choice and Welfare*, Vol. 2, K. J. Arrow, A. K. Sen, and K. Suzumura, eds. North-Holland, Amsterdam.

Barberà S., Beviá C., (2002). "Self-Selection Consistent Functions." *Journal of Economic Theory*, 105, 263-277.

Barberà S., Beviá C., (2005). "Locating Public Facilities by Majority: Stability, Consistency and Group formation." *Games and Economic Behavior*, 56, 185-200.

Barberà S., S., F. Gul, and E. Stacchetti (1993): "Generalized median voting schemes and committees", *Journal of Economic Theory* 61, 262–289.

Barberà S., Jackson M., (1994): "A Characterization of Strategy-Proof Social Choice Functions for Economies with Pure Public Goods." *Soc. Choice and Welfare*, 11, 241-252.

Barberà S., J. Massó , and A. Neme (1997): "Voting under constraints", *Journal of Economic Theory* 76, 298–321.

Barberà S., J. Massó , and A. Neme (2005): "Voting by committees under constraints", *Journal of Economic Theory* 122, 185–205.

Barberà S., J. Massó , and S. Serizawa (1998): "Strategy-proof voting on compact ranges", *Games and Economic Behavior* 25, 272–291.

Barberà S., H. Sonnenschein, and L. Zhou (1991): "Voting by committees", Econometrica 59, 595–609.

Black D., (1948). "On the Rationale of Group Decision-Making." *Journal of Political Economy*, 56, 23-34.

Bochet O., Gordon S., (2010). "Priorities in the Location of Multiple Public Facilities," CIREQ working paper, Université de Montréal,

https://www.webdepot.umontreal.ca/Usagers/gordons/MonDepotPublic/priorities.pdf

Bochet O., Gordon S., Saran R., (2009). "Generalized Majoritarian Rules for the Location of Multiple Public Facilities", Mimeo, Université de Montréal.

Bogomolnaia, A. Nicolò A., (2005). "Stable Assignment of Public Facilities under Congestion." *Journal of Public Economic Theory*, 7 (1), 65-91.

Border, K. C., and J. S. Jordan (1983): "Straightforward elections, unanimity and phantom voters", *Review of Economic Studies* 50, 153–170.

Cantala D. (2004). "Choosing the Level of a Public Good when Agents have an Outside Option." *Social Choice and Welfare*, 22, 491-514.

Ching S., (1997). "Strategy-Proofness and Median Voters." *International Journal of Game Theory*, 26, 473-490.

Ching S. and Thomson W., (1996). "Population-Monotonic Solutions in Public Good Economies with Single-Peaked Preferences." Forthcoming, *Social Choice and Welfare.*

Ehlers, L. (2001): "Independence Axioms for the Provision of Multiple Public Goods as Options", *Mathematical Social Sciences* 41-2, 239–250.

Ehlers L., (2002). "Multiple Public Goods and Lexicographic Preferences." *Journal of Mathematical Economics*, 37, 1-15.

Ehlers L., (2003). "Multiple Public Goods, Lexicographic Preferences, and Single-Plateaued Preference Rules." *Games and Economic Behavior*, 43, 1-27.

Ehlers, L., H. Peters and T. Storcken (2002): "Strategy-proof probabilistic decision schemes for one-dimensional single-peaked preferences", *Journal of Economic Theory,* 105, 408–434.

Gordon S., (2007a). "Public Decisions: Solidarity and the Status Quo." *Games and Economic Behavior*, 61, 225-241.

Gordon S., (2007b). "Solidarity in Choosing a Location on a Cycle." *Social Choice and Welfare*, 29, 125-147.

Heo, E. J. (2011) Strategy-proof Rules for Two Public Goods: Double Median Rules, Technical report, University of Rochester.

Jackson M., Nicolò A., (2004). "The Strategy-Proofness provision of Public Goods under Congestion and Crowding Preferences." *Journal of Economic Theory*, 115, 278-308.

Ju B., (2008). "Efficiency and Consistency for Locating Multiple Public Facilities." *Journal of Economic Theory*, 138, 165-183.

Le Breton, M., and A. Sen (1999): "Separable Preferences, Strategyproofness, and Decomposability", *Econometrica*, 67-3, 605–628.

Le Breton, M., and J. A. Weymark (1999) "Strategy-proof social choice with continuous separable preferences", *Journal of Mathematical Economics* 32, 47–85.

Miyagawa E., (1998). "Mechanisms for Providing a Menu of Public Goods." PhD dissertation, University of Rochester.

Miyagawa E., (2001). "Locating Libraries on a Street." *Soc. Choice and Welfare*, 18, 527-541.

Moulin, H. (1980): "On Strategy-Proofness and Single-Peakedness", *Public Choice 35*, 437–455.

Moulin, H. (1987): "The Pure Compensation Problem: Egalitarian versus Laissez-Fairism", *Quarterly Journal of Economics* 102, 769–783.

Schummer J. and R. Vohra (2002): "Strategy-proof Location on a Network", *Journal of Economic Theory* 104, 405–428.

Serizawa, S. (1999): "Strategy-proof and symmetric social choice functions for public good economies", *Econometrica*, 67, 121–145.

Thomson W., (1983a). "The Fair Division of a Fixed Supply among a Growing Population," *Mathematics of Operations Research*, 8, 319-326.

Thomson W., (1983b). "Problems of Fair Division and the Egalitarian Principle." *Journal of Economic Theory*, 31, 211-226.

Thomson, W. (1993): "The Replacement Principle in Public Good Economies with Single-Peaked Preferences", *Economics Letters 42*, 31–36.

Thomson, W., (1995). "Population-Monotonic Allocation Rules." In: Barrett W., Moulin H., Sallles M., Schofield N. (eds.), *Social Choice, Welfare, and Ethics*, Cambridge University Press, Cambridge, pp. 79-124.

Thomson, W., (1999a). "Welfare-domination under Preference-Replacement: a Survey and Open Questions." *Social Choice and Welfare*, 16, 373-394.

Thomson, W., (1999b). "Population-Monotonicity: a Survey and Open Questions." Working paper, University of Rochester.

Thomson, W. (1997b): "The Replacement Principle in Economies with Single-Peaked Preferences", *Journal of Economic Theory 76*, 145–168.

Weymark, J. A. (1999): "Decomposable strategy-proof social choice functions", *Japanese Economic Review* 50, 343–355.

Weymark, J. A. (2008): "Strategy-proofness and the Tops-only Property", *Journal of Public Economic Theory* 28, 7–26.

Figure 1: Typical parameters of a one agent *delta rule* in the case where $[\underline{a}, \overline{a}] \cap [\underline{b}, \overline{b}] = \emptyset$.

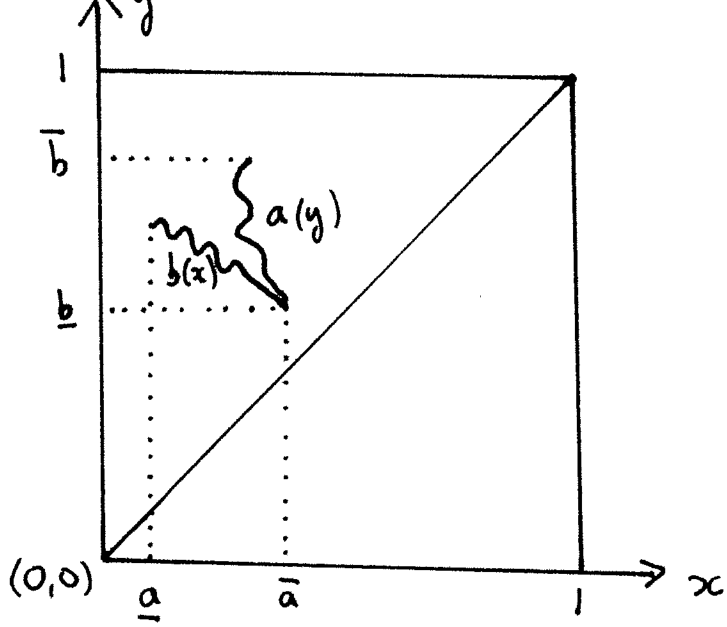

Figure 2: Typical parameters of a one agent *delta rule* in the case where $[\underline{a}, \overline{a}] \cap [\underline{b}, \overline{b}] \neq \emptyset$.

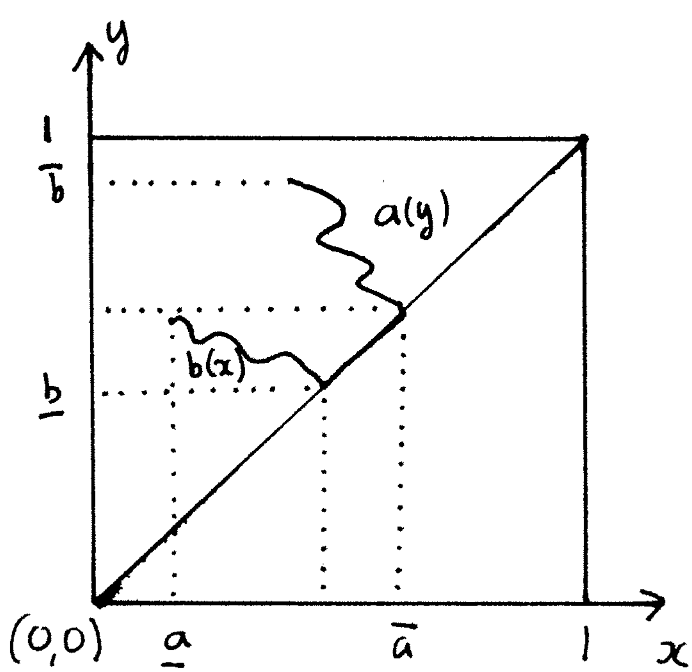

Figure 3: Typical parameters of a two agents *delta rule*.