

Additive representation of separable preferences over infinite products

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Abstract

Let \mathcal{X} be a set of outcomes, and let \mathcal{I} be an infinite indexing set. Any separable, permutation-invariant preference order (\succeq) on $\mathcal{X}^{\mathcal{I}}$ admits an additive representation. That is: there exists a linearly ordered abelian group \mathcal{R} and a ‘utility function’ $u : \mathcal{X} \rightarrow \mathcal{R}$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ which differ in only finitely many coordinates, we have $\mathbf{x} \succeq \mathbf{y}$ if and only if $\sum_{i \in \mathcal{I}} [u(x_i) - u(y_i)] \geq 0$. If (\succeq) also satisfies a weak continuity condition, then, for *any* $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \succeq \mathbf{y}$ if and only if ${}^*\sum_{i \in \mathcal{I}} u(x_i) \geq {}^*\sum_{i \in \mathcal{I}} u(y_i)$. Here, ${}^*\sum_{i \in \mathcal{I}} u(x_i)$ represents a nonstandard sum, taking values in a linearly ordered abelian group ${}^*\mathcal{R}$, which is an ultrapower extension of \mathcal{R} . These results are applicable to infinite-horizon intertemporal choice, choice under uncertainty, and variable-population social choice.

Keywords. additively separable preference; intertemporal choice; uncertainty; variable-population social choice; generalized utilitarian; nonstandard analysis; hyperreal; non-Archimedean utility

1 Model and main results

Let \mathcal{X} be a set of outcomes, and let \mathcal{I} be an infinite indexing set. An element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ assigns an outcome x_i to each $i \in \mathcal{I}$. The space $\mathcal{X}^{\mathcal{I}}$ has at least three interpretations:

- (i) *Intertemporal choice.* \mathcal{I} represents a ‘time-stream’ (an infinite sequence of moments in time e.g. $\mathcal{I} = \mathbb{N}$ or $\mathcal{I} = \mathbb{R}_+$) and \mathcal{X} represents the set of possible outcomes which could happen at each moment. An element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ thus represents a history where outcome x_i happens at time i .
- (ii) *Choice under uncertainty.* \mathcal{I} represents an infinite set of possible ‘states of nature’ (the true state is unknown), and \mathcal{X} is the set of possible outcomes which could occur in each state. An element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ thus represents a ‘lottery’ (or ‘Savage act’) which would yield the outcome x_i if state i occurs.¹

An important special case arises when $\mathcal{X} = \{0, 1\}$, so that there is an obvious bijection between $\mathcal{X}^{\mathcal{I}}$ and the set of all subsets of \mathcal{I} . In this case, a preorder on $\mathcal{X}^{\mathcal{I}}$ can also be interpreted as judging whether one subset of \mathcal{I} is ‘more probable’ than another.

¹Note that we do not assume a probability distribution on \mathcal{I} .

(iii) *Variable population social choice.* \mathcal{I} represents an infinite set of ‘potential people’, and \mathcal{X} represents the set of possible personal physical/psychological outcomes available to each person.² Suppose \mathcal{X} contains an outcome o representing ‘nonexistence’. If $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $x_i = o$ for all but finitely many coordinates, then \mathbf{x} represents a finite (but arbitrarily large) population.

One can also combine these interpretations:

(iv) *Variable population intertemporal social choice under uncertainty.* Let \mathcal{T} represent a time-stream (e.g. $\mathcal{T} := \mathbb{N}$), let \mathcal{S} be a set of possible ‘states of nature’, let \mathcal{P} be a set of ‘possible people’, and suppose at least one of \mathcal{T} , \mathcal{S} , or \mathcal{P} is infinite. Let $\mathcal{I} := \mathcal{T} \times \mathcal{S} \times \mathcal{P}$, and let \mathcal{X} be a space of personal psychophysical outcomes, including a ‘nonexistence’ outcome o . Then an element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ represents a policy which will assign personal outcome $x_{t,s,p}$ to person p at time t , if the state of nature s occurs.

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, let $\mathcal{I}(\mathbf{x}, \mathbf{y}) := \{i \in \mathcal{I}; x_i \neq y_i\}$, and $d(\mathbf{x}, \mathbf{y}) := |\mathcal{I}(\mathbf{x}, \mathbf{y})|$. A *finitary preorder* on $\mathcal{X}^{\mathcal{I}}$ is a transitive binary relation (\succeq) such that, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$,

$$\left(d(\mathbf{x}, \mathbf{y}) < \infty\right) \implies \left(\mathbf{x} \succeq \mathbf{y} \text{ or } \mathbf{x} \preceq \mathbf{y}\right). \quad (1)$$

(Note that (1) implies that (\succeq) is reflexive.) The relation (\succeq) is *strictly finitary* if the “ \implies ” in (1) is actually “ \iff ”.

A *linearly ordered abelian group* is a triple $(\mathcal{R}, +, >)$, where \mathcal{R} is a set, $+$ is an abelian group operation, and $>$ is a complete, antisymmetric, transitive binary relation such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$. For example: the set \mathbb{R} of real numbers is a linearly ordered abelian group (with the standard ordering and addition operator). So is any subgroup of \mathbb{R} . For any $n \in \mathbb{N}$, the space \mathbb{R}^n is a linearly ordered abelian group under vector addition and the lexicographic order. In fact, *Hahn’s Embedding Theorem* says that any linearly ordered abelian group can be represented as an ordered subgroup of a lexicographically ordered vector space (possibly infinite dimensional); see §5.1 for details.

Let $u : \mathcal{X} \rightarrow \mathcal{R}$ be a ‘utility function’. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $u(x_i) - u(y_i) = 0$ for all $i \in \mathcal{I} \setminus \mathcal{I}(\mathbf{x}, \mathbf{y})$. Thus, if $d(\mathbf{x}, \mathbf{y}) < \infty$, then

$$\sum_{i \in \mathcal{I}} \left(u(x_i) - u(y_i)\right) = \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} \left(u(x_i) - u(y_i)\right)$$

is a finite sum of elements in \mathcal{R} , and thus well-defined. One can then define a strictly finitary preorder ($\overset{u}{\succeq}$) on $\mathcal{X}^{\mathcal{I}}$ by specifying:

$$\left(\mathbf{x} \overset{u}{\succeq} \mathbf{y}\right) \iff \left(\sum_{i \in \mathcal{I}} \left(u(x_i) - u(y_i)\right) \geq 0\right), \quad (2)$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ with $d(\mathbf{x}, \mathbf{y}) < \infty$; this is the *additive preorder* induced by u . In interpretation (i) above, $\sum_{i \in \mathcal{I}} (u(x_i) - u(y_i))$ is the nondiscounted sum of future u -utility

²Thus, the elements of \mathcal{X} are ‘extended alternatives’, which encode both the specific identity of a person and any ethically relevant information about her physical and mental state.

differences between \mathbf{x} and \mathbf{y} . In interpretation (ii), the sum $\sum_{i \in \mathcal{I}} (u(x_i) - u(y_i))$ is the difference between the expected u -utility of lottery \mathbf{x} and that of lottery \mathbf{y} (assuming a uniform probability distribution on $\mathcal{I}(\mathbf{x}, \mathbf{y})$). In interpretation (iii), (\succeq_u) is a generalized utilitarian social welfare order. (See Example 4 below for further discussion.)

A preorder (\succeq) is *separable*³ if the following holds: for any $\mathcal{J} \subset \mathcal{I}$, with $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$, and for every $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$, if:

$$\begin{aligned} \mathbf{x}_{\mathcal{J}} &= \mathbf{y}_{\mathcal{J}}, & \mathbf{x}_{\mathcal{K}} &= \mathbf{x}'_{\mathcal{K}}, \\ \mathbf{x}'_{\mathcal{J}} &= \mathbf{y}'_{\mathcal{J}}, & \mathbf{y}_{\mathcal{K}} &= \mathbf{y}'_{\mathcal{K}}, \end{aligned} \tag{3}$$

then $(\mathbf{x} \succeq \mathbf{y}) \iff (\mathbf{x}' \succeq \mathbf{y}')$. Heuristically: if $\mathbf{x}_{\mathcal{J}} = \mathbf{y}_{\mathcal{J}}$, then the ordering between \mathbf{x} and \mathbf{y} should be decided entirely by comparing $\mathbf{x}_{\mathcal{K}}$ and $\mathbf{y}_{\mathcal{K}}$. Likewise, if $\mathbf{x}'_{\mathcal{J}} = \mathbf{y}'_{\mathcal{J}}$, then the ordering between \mathbf{x}' and \mathbf{y}' should be decided entirely by comparing $\mathbf{x}'_{\mathcal{K}}$ and $\mathbf{y}'_{\mathcal{K}}$. Thus, if $\mathbf{x}_{\mathcal{K}} = \mathbf{x}'_{\mathcal{K}}$ and $\mathbf{y}_{\mathcal{K}} = \mathbf{y}'_{\mathcal{K}}$, then the ordering between \mathbf{x} and \mathbf{y} should agree with the ordering between \mathbf{x}' and \mathbf{y}' .

A *finitary permutation* is a bijection $\pi : \mathcal{I} \rightarrow \mathcal{I}$ such that the set $\mathcal{I}(\pi) := \{i \in \mathcal{I}; \pi(i) \neq i\}$ is finite. Let Π_{fin} be the group of all finitary permutations of \mathcal{I} . For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi_{\text{fin}}$, we have $\mathcal{I}(\mathbf{x}, \pi(\mathbf{x})) \subseteq \mathcal{I}(\pi)$, so $d(\mathbf{x}, \pi(\mathbf{x})) < \infty$, so \mathbf{x} is comparable to $\pi(\mathbf{x})$. Say that (\succeq) is Π_{fin} -*invariant* if $\mathbf{x} \approx \pi(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi_{\text{fin}}$. In interpretation (i) above, Π_{fin} -invariance means there are no time preferences: the near and far future are equally important. In interpretation (ii), Π_{fin} -invariance means that all states of nature are regarded as equally likely. In interpretation (iii), Π_{fin} -invariance translates into ‘anonymity’: all people must be treated the same by the social preference relation (\succeq) . In interpretation (iv), Π_{fin} -invariance implies all three of these things. Here is our first main result.

Theorem 1 *Let (\succeq) be a strictly finitary preorder on $\mathcal{X}^{\mathcal{I}}$. Then (\succeq) is Π_{fin} -invariant and separable if and only if there exists some linearly ordered abelian group $(\mathcal{R}, +, >)$ and function $u : \mathcal{X} \rightarrow \mathcal{R}$ such that (\succeq) is the additive preorder defined by u .*

Furthermore, \mathcal{R} and u can be built with the following universal property: if $(\mathcal{R}', +, >)$ is another linearly ordered abelian group, and (\succeq) is also the additive preorder defined by some function $u' : \mathcal{X} \rightarrow \mathcal{R}'$, then there exists some $r' \in \mathcal{R}'$ and some order-preserving group homomorphism $\psi : \mathcal{R} \rightarrow \mathcal{R}'$ such that $u'(x) = \psi[u(x)] + r'$ for all $x \in \mathcal{X}$.

The proof of Theorem 1 uses (\succeq) to construct a complete preorder on the free abelian group \mathcal{A} generated by \mathcal{X} (the infinite cardinality of \mathcal{I} is essential here). We then factor \mathcal{A} by the indifference relation to obtain \mathcal{R} . This strategy is somewhat analogous to Wakker [58]. The proof of Theorem 1 and all other results are in the Appendix.

Theorem 1 is sufficient for variable-population social choice, or even for intertemporal (or uncertain) choices between social alternatives which differ at only finitely many moments in time (or states of nature). However, it is not sufficient for choice problems which implicate infinitely many coordinates. To extend Theorem 1 to this setting, we will use nonstandard analysis. Given any linearly ordered abelian group \mathcal{R} , one can construct a

³In Savage’s theory, this property is called the *sure thing principle* or Axiom P2. In axiomatic measurement theory, it is variously called (*joint*) *independence* or *single cancellation*. In social choice, separability is a special case of the axiom of *independence of* (or *elimination of*) *indifferent individuals*, which in turn is a special case of the *Extended Pareto* axiom.

larger linearly ordered abelian group ${}^*\mathcal{R}$ by supplementing \mathcal{R} with a rich collection of ‘infinite’ and ‘infinitesimal’ elements with their own well-defined arithmetic.⁴ (For example, if \mathcal{R} is the additive group \mathbb{R} of real numbers, then ${}^*\mathbb{R}$ is the additive group of *hyperreal* numbers.) For any function $u : \mathcal{X} \rightarrow \mathcal{R}$ and any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, it is possible to evaluate the ‘sum’ ${}^*\sum_{i \in \mathcal{I}} u(x_i)$ as an element of ${}^*\mathcal{R}$ in a unique and well-defined way. We can then define the *hyperadditive* preorder $(\stackrel{*}{\succeq}_u)$ on $\mathcal{X}^{\mathcal{I}}$ by

$$\left(\mathbf{x} \stackrel{*}{\succeq}_u \mathbf{y} \right) \iff \left({}^*\sum_{i \in \mathcal{I}} u(x_i) \geq {}^*\sum_{i \in \mathcal{I}} u(y_i) \right), \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}. \quad (4)$$

$(\stackrel{*}{\succeq}_u)$ is a *complete*, Π_{fin} -invariant, separable preorder on $\mathcal{X}^{\mathcal{I}}$, whose finitary part is the additive preorder $(\stackrel{\sim}{\succeq}_u)$. Furthermore, it satisfies a weak continuity condition called $\mathcal{U}\mathfrak{F}$ -*continuity*, which will be formally defined in §3. (Roughly: $\mathcal{U}\mathfrak{F}$ -continuity says that $\mathbf{x} \succeq \mathbf{y}$ if $\mathbf{x}_{\mathcal{J}} \succeq \mathbf{y}_{\mathcal{J}}$ for ‘almost all’ finite subsets $\mathcal{J} \subset \mathcal{I}$.) Our second main result says that a subset of these properties suffice to characterize hyperadditive preorders.

Theorem 2 *Let (\succeq) be a preorder on $\mathcal{X}^{\mathcal{I}}$. Then (\succeq) is Π_{fin} -invariant, separable and $\mathcal{U}\mathfrak{F}$ -continuous if and only if there exists some linearly ordered abelian group $(\mathcal{R}, +, >)$ and some function $u : \mathcal{X} \rightarrow \mathcal{R}$ such that $(\succeq) = (\stackrel{*}{\succeq}_u)$.*

Furthermore, \mathcal{R} and u can be built with the same universal property as in Theorem 1.

In what sense does the function u represent individual preferences in Theorems 1 and 2? For any $x \in \mathcal{X}$, any $i \in \mathcal{I}$, and any $\mathbf{z}_{-i} \in \mathcal{X}^{\mathcal{I} \setminus \{i\}}$, let (x, \mathbf{z}_{-i}) be the element of $\mathcal{X}^{\mathcal{I}}$ which has x in the i th coordinate and the entries of \mathbf{z}_{-i} in the other coordinates. Given a finitary preorder (\succeq) on $\mathcal{X}^{\mathcal{I}}$, we define a complete preorder $(\stackrel{\sim}{\succeq}_1)$ on \mathcal{X} as follows: For any $x, y \in \mathcal{X}$, define $x \stackrel{\sim}{\succeq}_1 y$ if there exists $i \in \mathcal{I}$ and $\mathbf{z}_{-i} \in \mathcal{X}^{\mathcal{I} \setminus \{i\}}$ such that $(x, \mathbf{z}_{-i}) \succeq (y, \mathbf{z}_{-i})$. If (\succeq) is separable and Π_{fin} -invariant, then it follows that $x \stackrel{\sim}{\succeq}_1 y$ if and only if $(x, \mathbf{z}_{-i}) \succeq (y, \mathbf{z}_{-i})$ for all $i \in \mathcal{I}$ and $\mathbf{z}_{-i} \in \mathcal{X}^{\mathcal{I} \setminus \{i\}}$. The proof of the next result is straightforward.

Proposition 3 *Let $(\mathcal{R}, +, >)$ be a linearly ordered group, let $u : \mathcal{X} \rightarrow \mathcal{R}$, and suppose (\succeq) is either the additive or hyperadditive preorder on $\mathcal{X}^{\mathcal{I}}$ defined by u . Then u is a utility function for $(\stackrel{\sim}{\succeq}_1)$. Furthermore, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ which are (\succeq) -comparable, we have:*

$$\left(\begin{array}{l} x_i \stackrel{\sim}{\succeq}_1 y_i \text{ for all } i \in \mathcal{I} \\ x_i \stackrel{\sim}{\succeq}_1 y_i \text{ for all } i \in \mathcal{I}, \text{ and } x_i \stackrel{\sim}{\succeq}_1 y_i \text{ for some } i \in \mathcal{I} \end{array} \right) \implies \left(\begin{array}{l} \mathbf{x} \succeq \mathbf{y} \\ \mathbf{x} \succ \mathbf{y} \end{array} \right). \quad (5)$$

Example 4. (i) Under interpretation (i), statement (5) is sometimes called *monotonicity*, while Π_{fin} -invariance implies no time preferences. In this case, u is a single-period utility function for the individual or population in question, and Theorem 1 is comparable to recent characterizations of ‘intertemporal utilitarianism’ by Banerjee [6] and Basu and Mitra [10] (see §5.2 for more details). Meanwhile, Theorem 2 is similar to Theorem 5 of

⁴Formally, ${}^*\mathcal{R}$ is an ultrapower of \mathcal{R} with respect to an ultrafilter $\mathcal{U}\mathfrak{F}$ defined over the set of all finite subsets of \mathcal{I} . The precise construction of ${}^*\mathcal{R}$ is somewhat technical, and will be provided in §3 below.

Fleurbaey and Michel [29]. These models all assume $\mathcal{X} = \mathbb{R}$ (with standard ordering) and $\mathcal{I} = \mathbb{N}$. Banerjee [6] and Basu and Mitra [10] do not utilize the ordering of \mathbb{N} , but Fleurbaey and Michel's [29] $\ast\mathbb{R}$ -valued SWF on $\mathbb{R}^{\mathbb{N}}$ does: it is defined in terms of sums over the sets $[1 \dots T]$ for $T \in \mathbb{N}$.

(ii) Under interpretation (ii), statement (5) is the *statewise dominance* axiom. In this case, u can be seen as a Bernoulli cardinal utility function, and $\ast\sum_{i \in \mathcal{I}} u(x_i)$ represents the 'expected utility' of the lottery \mathbf{x} . Thus, Theorems 1 and 2 are comparable to Savage's [53] model of subjective expected utility, and its more recent lexicographic extensions (see §5.1 below). Separability corresponds to Savage's axiom P2 (the *Sure Thing Principle*), while Π_{fin} -invariance implies Savage's P3 (*state-independent ordinal preferences*).

However, Savage [53] also assumes that (\succeq) is 'nonatomic'; hence all finite subsets of \mathcal{I} are null. In contrast, every nonempty subset of \mathcal{I} is non-null for $(\frac{\succeq}{u})$ and $(\frac{\ast\sum}{u})$. Also, in Savage's model, the utility function u is bounded, whereas in Theorems 1 and 2 it might not be. Finally, Savage's model yields not only u , but a finitely additive, \mathbb{R} -valued 'subjective' probability measure on \mathcal{I} ; both depend on (\succeq) . In contrast, Π_{fin} -invariance acts like Laplace's 'Principle of insufficient reason', and effectively determines a finitely additive, 'uniformly distributed' $\ast\mathbb{N}$ -valued measure on \mathcal{I} , independent of (\succeq) (see Example 8 below). Thus, (\succeq) is entirely determined by u alone in Theorems 1 and 2. Indeed, any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ induces a finitely additive $\ast\mathbb{N}$ -valued measure $\ast\rho$ on \mathcal{X} , such that $\ast\sum_{i \in \mathcal{I}} u(x_i) = \sum_{x \in \mathcal{X}} \ast\rho(x) u(x)$. Given this correspondence, Theorems 1 and 2 are perhaps more comparable to the von Neumann-Morgenstern model than the Savage model.

(iii) Under interpretation (iii), statement (5) is the *strong Pareto* axiom, while Π_{fin} -invariance is *anonymity*. In this case, u can be seen as a measure of individual well-being, and $\ast\sum_{i \in \mathcal{I}} u(x_i)$ represents the 'aggregate utility' of a social policy \mathbf{x} for society. Thus, Theorems 1 and 2 are comparable to Blackorby *et al.*'s [11, Thm.2] variable-population characterization of generalized utilitarianism, or Fleming's [28] and Debreu's [23] fixed-population characterizations of generalized utilitarianism.⁵

(iv) Suppose we combine interpretations (ii) and (iii). Let \mathcal{P} be a space of possible people, let \mathcal{S} be an infinite space of states of nature, and let $\mathcal{I} = \mathcal{P} \times \mathcal{S}$. Thus, an element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ represents a 'social lottery' which assigns outcome $x_{p,s} \in \mathcal{X}$ to person $p \in \mathcal{P}$ in state of nature $s \in \mathcal{S}$. For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $p \in \mathcal{P}$, let $\mathbf{x}_p := (x_{p,s})_{s \in \mathcal{S}}$ (an element of $\mathcal{X}^{\mathcal{S}}$); this is the 'personal lottery' which \mathbf{x} induces for person p . Define $\bar{u}(\mathbf{x}_p) := \ast\sum_{s \in \mathcal{S}} u(x_{p,s})$ —the expected utility of \mathbf{x}_p . Then $\ast\sum_{i \in \mathcal{I}} u(x_{p,s}) = \ast\sum_{p \in \mathcal{P}} \bar{u}(\mathbf{x}_p)$. Thus, $\mathbf{x} \frac{\ast\sum}{u} \mathbf{y}$ if and only if $\ast\sum_{p \in \mathcal{P}} \bar{u}(\mathbf{x}_p) \geq \ast\sum_{p \in \mathcal{P}} \bar{u}(\mathbf{y}_p)$. In this setting, Theorems 1 and 2 closely resemble Harsanyi's [35] social aggregation theorem.

(v) (*Leximin order*) Suppose $\mathcal{X} = \mathbb{R}$. Let $\mathbb{Z}^{\mathbb{R}}$ denote the set of all functions from \mathbb{R} into \mathbb{Z} , and let $\mathcal{R} := \{\mathbf{z} \in \mathbb{Z}^{\mathbb{R}}; z_r \neq 0 \text{ for only finitely many } r \in \mathbb{R}\}$. Let $+$ be the operation of componentwise addition on \mathcal{R} , and let $(>)$ be the lexicographical order on \mathcal{R} ; then $(\mathcal{R}, +, >)$ is a linearly ordered abelian group. For any $x \in \mathbb{R}$, let $u(x)$ denote the element

⁵'Generalized' because u might actually be a monotone increasing transformation of the 'true' cardinal utility function of the individuals.

$\mathbf{z} \in \mathbb{Z}^{\mathbb{R}}$ such that $z_x = -1$, while $z_y = 0$ for all $y \in \mathbb{R} \setminus \{x\}$; this defines a function $u : \mathbb{R} \rightarrow \mathcal{R}$. The additive preorder $(\underset{u}{\succeq})$ is the (finitary) leximin preorder on $\mathbb{R}^{\mathcal{I}}$. \diamond

The rest of this paper is organized as follows. Section 2 discusses when u is real-valued. Section 3 formally defines the ultrapower group ${}^*\mathcal{R}$ and the hyperadditive order $(\underset{u}{\succ})$, and states a series of lemmas which are the main steps in the proof of Theorem 2. Section 4 examines the meaning of the three axioms invoked in Theorems 1 and 2. Section 5 briefly reviews related literature. Finally, the Appendix contains the proofs of all results.

2 Archimedean utilities

In Theorems 1 and 2, when can the utility function u be treated as real-valued? (Equivalently: when is $\mathcal{R} \subseteq \mathbb{R}$?) A linearly ordered group \mathcal{R} is *Archimedean* if, for any $r, s \in \mathcal{R}$ with $r > 0$, there exists some $N \in \mathbb{N}$ such that $N \cdot r \geq s$. Heuristically, this means that \mathcal{R} contains no ‘infinite’ or ‘infinitesimal’ elements. For example: the additive group $(\mathbb{R}, +)$ (with the standard ordering) is Archimedean. But if $n \geq 2$, then the lexicographically ordered group \mathbb{R}^n is *not* Archimedean. Nor is the additive group ${}^*\mathbb{R}$ of hyperreal numbers.

Hölder’s theorem. *A linearly ordered group $(\mathcal{R}, +, >)$ is Archimedean if and only if there is an order-preserving group isomorphism from \mathcal{R} into some subgroup of $(\mathbb{R}, +)$.*

Fix $o \in \mathcal{X}$ and define $\mathbf{o} \in \mathcal{X}^{\mathcal{I}}$ by $o_i := o$ for all $i \in \mathcal{I}$. For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ with $d(\mathbf{x}, \mathbf{o}) < \infty$, and any $N \in \mathbb{N}$, define \mathbf{x}^N as follows: find disjoint subsets $\mathcal{J}_1, \dots, \mathcal{J}_N \subset \mathcal{I}$ with $|\mathcal{J}_n| = d(\mathbf{x}, \mathbf{o})$ for all $n \in [1 \dots N]$, let $\beta_n : \mathcal{J}_n \rightarrow \mathcal{I}(\mathbf{x}, \mathbf{o})$ be bijections for $n \in [1 \dots N]$, and then define $x_j^N := x_{\beta_n(j)}$ for all $n \in [1 \dots N]$ and $j \in \mathcal{J}_n$, whereas $x_i^N := o$ for all $i \in \mathcal{I} \setminus \mathcal{J}_1 \sqcup \dots \sqcup \mathcal{J}_N$. We say (\succeq) is *o-Archimedean* if and only if: for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ with $d(\mathbf{x}, \mathbf{o}) < \infty$, $d(\mathbf{y}, \mathbf{o}) < \infty$, and $\mathbf{x} \succ \mathbf{o}$, there exists some $N \in \mathbb{N}$ such that $\mathbf{x}^N \succeq \mathbf{y}$. If (\succeq) is separable, then this definition is independent of the choice of reference element $o \in \mathcal{X}$; in this case we simply say (\succeq) is *Archimedean*.⁶

Corollary 5 *Let (\succeq) be a strictly finitary preorder on $\mathcal{X}^{\mathcal{I}}$. Then (\succeq) is Π_{fin} -invariant, separable, and Archimedean if and only if there exists some $u : \mathcal{X} \rightarrow \mathbb{R}$ such that (\succeq) is the additive preorder defined by u .⁷*

Suppose $x \in \mathcal{X}$ represents an outcome which is ‘just barely’ better than o . Let (x^N, \mathbf{o}) denote an element of $\mathcal{X}^{\mathcal{I}}$ which has x in exactly N coordinates, and o in all other coordinates. If (\succeq) is an o -Archimedean preorder on $\mathcal{X}^{\mathcal{I}}$, then interpretation (iii) yields what Parfit [47] calls the *Repugnant Conclusion*: for any $\mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ with $d(\mathbf{y}, \mathbf{o}) < \infty$, there exists some $N \in \mathbb{N}$ such that $(x^N, \mathbf{o}) \succ \mathbf{y}$. (See Ryberg and Tannsjo [50] and also Blackorby *et al.* [12, §6] and [13] for further discussion.)

On the other hand, if u is a non-Archimedean utility function (indeed, even if u Archimedean, but unbounded above), then there is an opposite but equally repugnant

⁶In different contexts, the Archimedean property has been called *continuity* or *substitutability*.

⁷In the case $\mathcal{I} = \mathbb{N}$, Corollary 5 is equivalent to Theorem 1.1 of Wakker [58], which in turn builds on earlier, similar results by Camacho [17, 18, 19, 20].

conclusion. In this case, there exists $x \in \mathcal{X}$ with $u(x) > 0$, such that, for any $N \in \mathbb{N}$, there exists some $y \in \mathcal{X}$ such that $N \cdot u(x) < u(y)$, and thus, $(x^N, \mathbf{o}) \prec (y^1, \mathbf{o})$. (If u is non-Archimedean, there exists a single $y \in \mathcal{X}$ which works for all $N \in \mathbb{N}$). In interpretation (iii), x represents an outcome of reasonable happiness or satisfaction, while y represents Nozick's 'utility monster': (\succeq) says it is better to starve N moderately happy people, just so that one person can achieve the 'Nirvana' outcome y . Interpretation (ii) yields the St. Petersburg Paradox: (x^N, \mathbf{o}) represents the status quo, and (y_N^1, \mathbf{o}) represents a gamble with a very small ($\frac{1}{N}$) probability of yielding a fabulous payoff y_N , and a very large $(1 - \frac{1}{N})$ probability of total ruin. Interpretation (i) yields the 'Paradox of Eternal Reinvestment'. There are similar paradoxes if u is unbounded below. To avoid them, u must be both Archimedean and **bounded**: for any $x, o \in \mathcal{X}$, if $u(x) - u(o) > 0$, then there is some $M \in \mathbb{N}$ such that $-M [u(x) - u(o)] < u(y) - u(o) < M [u(x) - u(o)]$ for all $y \in \mathcal{X}$.

In general, \mathcal{R} will not be Archimedean; even if \mathcal{R} is, its ultrapower ${}^*\mathcal{R}$ certainly will not be. The literature on non-Archimedean utilities is reviewed in §5.1 below.

3 Formal definition of $(\overset{*}{\sum}_u)$

Let \mathcal{F} be the set of all finite subsets of \mathcal{I} . Elements of \mathcal{F} (i.e. subsets of \mathcal{I}) will be indicated by caligraphic letters (\mathcal{J}, \mathcal{K}). Let \mathfrak{P} be the power set of \mathcal{F} . Typical elements of \mathfrak{P} (i.e. subsets of \mathcal{F}) will be denoted by bold caligraphic letters (e.g. $\mathcal{D}, \mathcal{E}, \mathcal{G}$). Subsets of \mathfrak{P} are upper-case Fraktur (e.g. $\mathfrak{F}, \mathfrak{U}\mathfrak{F}$). A *free filter* is a subset $\mathfrak{F} \subset \mathfrak{P}$ with the following properties:

(F0) No finite subset of \mathcal{F} is an element of \mathfrak{F} . (In particular, $\emptyset \notin \mathfrak{F}$.)

(F1) If $\mathcal{D}, \mathcal{E} \in \mathfrak{F}$, then $\mathcal{D} \cap \mathcal{E} \in \mathfrak{F}$.

(F2) For any $\mathcal{E} \in \mathfrak{F}$ and $\mathcal{P} \in \mathfrak{P}$, if $\mathcal{E} \subseteq \mathcal{P}$, then $\mathcal{P} \in \mathfrak{F}$.

For any $\mathcal{P} \in \mathfrak{P}$, axioms (F0) and (F1) together imply that at most one of \mathcal{P} or \mathcal{P}^c can be in \mathfrak{F} . A *free ultrafilter* is free filter $\mathfrak{U}\mathfrak{F} \subset \mathfrak{P}$ which also satisfies:

(UF) For any $\mathcal{P} \in \mathfrak{P}$, either $\mathcal{P} \in \mathfrak{U}\mathfrak{F}$ or $\mathcal{P}^c \in \mathfrak{U}\mathfrak{F}$.

Equivalently, $\mathfrak{U}\mathfrak{F}$ a 'maximal' free filter: it is not a proper subset of any other free filter. Heuristically, elements of $\mathfrak{U}\mathfrak{F}$ are 'large' collections of finite subsets of \mathcal{I} ; if $\mathcal{G} \in \mathfrak{U}\mathfrak{F}$ and a certain statement holds for all $\mathcal{G} \in \mathcal{G}$, then this statement holds for 'almost all' finite subsets of \mathcal{I} . In particular, axioms (F0) and (UF) imply that $\mathcal{F} \in \mathfrak{U}\mathfrak{F}$. Given the Axiom of Choice, the existence of free ultrafilters is assured by the following result:

Ultrafilter lemma. *Every free filter \mathfrak{F} is contained in some free ultrafilter.*

Proof sketch. Consider the set of all free filters containing \mathfrak{F} ; apply Zorn's Lemma to get a maximal element of this set. □

Let $(\mathcal{R}, +, >)$ be a linearly ordered abelian group (e.g. $\mathcal{R} = \mathbb{R}$). Let $\mathcal{R}^{\mathcal{F}}$ be the set of all functions $r : \mathcal{F} \rightarrow \mathcal{R}$. For any $r, s \in \mathcal{R}^{\mathcal{F}}$, let $\mathcal{F}(r, s) := \{\mathcal{F} \in \mathcal{F}; r(\mathcal{F}) \geq s(\mathcal{F})\}$, and then define $r \succ_{\mathcal{U}\mathfrak{F}} s$ if and only if $\mathcal{F}(r, s) \in \mathcal{U}\mathfrak{F}$. Then $(\succ_{\mathcal{U}\mathfrak{F}})$ is a complete preorder on $\mathcal{R}^{\mathcal{F}}$. (Proof: $(\succ_{\mathcal{U}\mathfrak{F}})$ is complete by Axiom (UF). Next, $(\succ_{\mathcal{U}\mathfrak{F}})$ is reflexive, because $\mathcal{F} \in \mathcal{U}\mathfrak{F}$. Finally $(\succ_{\mathcal{U}\mathfrak{F}})$ is transitive, by Axioms (F1) and (F2).) Let $(\approx_{\mathcal{U}\mathfrak{F}})$ be the symmetric part of $(\succ_{\mathcal{U}\mathfrak{F}})$. Then $(\approx_{\mathcal{U}\mathfrak{F}})$ is an equivalence relation on $\mathcal{R}^{\mathcal{F}}$. We define ${}^*\mathcal{R} := \mathcal{R}^{\mathcal{F}} / (\approx_{\mathcal{U}\mathfrak{F}})$. For any $r \in \mathcal{R}^{\mathcal{F}}$, let *r denote the equivalence class of r in ${}^*\mathcal{R}$.⁸

The preorder $(\succ_{\mathcal{U}\mathfrak{F}})$ factors to a linear order $(>)$ on ${}^*\mathcal{R}$, defined by $({}^*r > {}^*s) \iff (r \succ_{\mathcal{U}\mathfrak{F}} s)$ for all ${}^*r, {}^*s \in {}^*\mathcal{R}$. Note that $\mathcal{R}^{\mathcal{F}}$ is an abelian group under pointwise addition. We can define a binary operation ‘+’ on ${}^*\mathcal{R}$ by setting ${}^*r + {}^*s := {}^*(r + s)$ for all ${}^*r, {}^*s \in {}^*\mathcal{R}$.

Lemma 6 $({}^*\mathcal{R}, +, >)$ is a linearly ordered abelian group.⁹

For any $r \in \mathcal{R}$, define $\bar{r} \in \mathcal{R}^{\mathcal{F}}$ by setting $\bar{r}(\mathcal{F}) := r$ for all $\mathcal{F} \in \mathcal{F}$. Then define $r' := {}^*\bar{r} \in {}^*\mathcal{R}$. Let $\mathcal{R}' := \{r'; r \in \mathcal{R}\} \subset {}^*\mathcal{R}$; then \mathcal{R}' is an ordered subgroup of ${}^*\mathcal{R}$, and it is isomorphic to \mathcal{R} via the map $r \mapsto r'$. We can thus identify \mathcal{R}' with \mathcal{R} , and regard ${}^*\mathcal{R}$ as an ordered group obtained by enriching \mathcal{R} with ‘infinite’ and ‘infinitesimal’ quantities.¹⁰

Example 7. Suppose \mathcal{I} is countable. Then \mathcal{F} is also countable; hence $\mathcal{R}^{\mathcal{F}}$ can be identified with $\mathcal{R}^{\mathbb{N}}$, and $\mathcal{U}\mathfrak{F}$ can be seen as an ultrafilter on \mathbb{N} . If $\mathcal{R} = \mathbb{R}$, then ${}^*\mathcal{R} = {}^*\mathbb{R}$ is the *hyperreal numbers* (seen as a linearly ordered abelian group); see Anderson [1]. If $\mathcal{R} \subset \mathbb{R}$, then ${}^*\mathcal{R}$ is some subgroup of ${}^*\mathbb{R}$. \diamond

Fix a function $r : \mathcal{I} \rightarrow \mathcal{R}$. For any $\mathcal{F} \in \mathcal{F}$, define $S(\mathcal{F}) := \sum_{f \in \mathcal{F}} r_f$. This yields a function $S : \mathcal{F} \rightarrow \mathcal{R}$. Define ${}^*\sum_{i \in \mathcal{I}} r_i := {}^*S$ (the equivalence class of S in ${}^*\mathcal{R}$). In particular, for any set \mathcal{X} , any function $u : \mathcal{X} \rightarrow \mathcal{R}$ and any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, define ${}^*\sum_{i \in \mathcal{I}} u(x_i) \in {}^*\mathcal{R}$ in this way. Then define the *hyperadditive* preorder (\succ_u) by formula (4).

Example 8. (*Uniform probability measure*) Let \mathbb{Z} denote the ring of integers; then \mathcal{R} is a \mathbb{Z} -module. Let ${}^*\mathbb{Z}$ denote the ultrapower of \mathbb{Z} modulo \mathcal{F} . Then ${}^*\mathbb{Z}$ is an integral domain, and ${}^*\mathcal{R}$ is a ${}^*\mathbb{Z}$ -module, by an argument similar to Lemma 6.

For any $\mathcal{J} \subseteq \mathcal{I}$, let $\mathbf{1}_{\mathcal{J}} : \mathcal{I} \rightarrow \mathbb{Z}$ be the indicator function of \mathcal{J} (i.e. $\mathbf{1}_{\mathcal{J}}(i) := 1$ if $i \in \mathcal{J}$, while $\mathbf{1}_{\mathcal{J}}(i) := 0$ if $i \notin \mathcal{J}$). If $\mathcal{J} \subseteq \mathcal{I}$ is finite, then ${}^*\sum_{i \in \mathcal{I}} \mathbf{1}_{\mathcal{J}}(i) = |\mathcal{J}|$. However, ${}^*\mu(\mathcal{J}) := {}^*\sum_{i \in \mathcal{I}} \mathbf{1}_{\mathcal{J}}(i) \in {}^*\mathbb{Z}$ is also well-defined when \mathcal{J} is infinite, and gauges the ‘size’ of the set \mathcal{J} as a subset of \mathcal{I} . In fact, ${}^*\mu$ is a finitely additive, ${}^*\mathbb{Z}$ -valued ‘probability measure’ defined on all subsets of \mathcal{I} .

For any function $r : \mathcal{I} \rightarrow \mathcal{R}$, the ${}^*\mathcal{R}$ -valued sum ${}^*\sum_{i \in \mathcal{I}} r(i)$ can be interpreted as the ‘integral’ of r with respect to the measure ${}^*\mu$. In particular, for any set \mathcal{X} , an element

⁸Formally, ${}^*\mathcal{R}$ is called the *ultrapower* of \mathcal{R} modulo the ultrafilter $\mathcal{U}\mathfrak{F}$. It is conventional to denote ultrapower-related objects with the leading star $*$.

⁹In fact, Lemma 6 is a special case of Łoś’s theorem, which roughly states that any first-order properties of any system of algebraic structures and/or N -ary relations on \mathcal{R} are ‘inherited’ by ${}^*\mathcal{R}$. For example, if \mathcal{R} is a linearly ordered field, then ${}^*\mathcal{R}$ will also be a linearly ordered field.

¹⁰A positive element of ${}^*\mathcal{R}$ is ‘infinitesimal’ if it is smaller than every positive element of \mathcal{R}' . It is ‘infinite’ if it is bigger than every element of \mathcal{R}' .

$\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ can be interpreted as an \mathcal{X} -valued random variable over the probability space $(\mathcal{I}, * \mu)$. The sum $* \sum_{i \in \mathcal{I}} u(x_i)$ is the ‘expected utility’ of \mathbf{x} . Thus, $(\overset{*}{\sum}_u)$ ranks elements of $\mathcal{X}^{\mathcal{I}}$ according to their $* \mu$ -expected u -utility.¹¹ \diamond

Lemma 9 $(\overset{*}{\sum}_u)$ is a complete, separable preorder on $\mathcal{X}^{\mathcal{I}}$.

Let Π be the group of all permutations of \mathcal{I} . Any $\pi \in \Pi$ defines a bijection $\pi : \mathcal{F} \rightarrow \mathcal{F}$ (because for any $\mathcal{J} \subseteq \mathcal{I}$, the image $\pi(\mathcal{J})$ is finite if and only if \mathcal{J} is finite). Define

$$\mathcal{F}(\pi) := \{\mathcal{F} \in \mathcal{F} ; \pi(\mathcal{F}) = \mathcal{F}\} = \{\mathcal{F} \subseteq \mathcal{I} ; \mathcal{F} \text{ a finite disjoint union of finite } \pi\text{-orbits}\}.$$

For example, if $\pi \in \Pi_{\text{fin}}$, so that $\mathcal{I}(\pi) := \{i \in \mathcal{I} ; \pi(i) \neq i\}$ is finite, then any finite superset of $\mathcal{I}(\pi)$ is an element of $\mathcal{F}(\pi)$. Let $\Pi_{\mathcal{U}\mathfrak{F}} := \{\pi \in \Pi ; \mathcal{F}(\pi) \in \mathcal{U}\mathfrak{F}\}$. (Heuristically, the elements of $\Pi_{\mathcal{U}\mathfrak{F}}$ are ‘measure-preserving’ transformations of \mathcal{I} . If $* \mu$ is the uniform measure from Example 8, then $* \mu[\pi(\mathcal{J})] = * \mu[\mathcal{J}]$ for every $\mathcal{J} \subseteq \mathcal{I}$ and $\pi \in \Pi_{\mathcal{U}\mathfrak{F}}$.) Let (\succeq) be a preorder on $\mathcal{X}^{\mathcal{I}}$. Say (\succeq) is $\Pi_{\mathcal{U}\mathfrak{F}}$ -invariant if $\mathbf{x} \approx \pi(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi_{\mathcal{U}\mathfrak{F}}$.

Lemma 10 (a) $\Pi_{\mathcal{U}\mathfrak{F}}$ is a subgroup of Π .

(b) $(\overset{*}{\sum}_u)$ is $\Pi_{\mathcal{U}\mathfrak{F}}$ -invariant.

Everything so far applies to any ultrafilter $\mathcal{U}\mathfrak{F} \subseteq \mathfrak{F}$. Now we will focus on a specific class of ultrafilters. Let $\Gamma \subset \Pi$ be a subgroup of Π . For any $i \in \mathcal{I}$, let $\mathcal{O}_{\Gamma}(i) := \{\gamma(i) ; \gamma \in \Gamma\}$ be the Γ -orbit of i . The orbit partition of Γ is the collection $\mathcal{O}_{\Gamma} := \{\mathcal{O}_{\Gamma}(i) ; i \in \mathcal{I}\}$. Say Γ has finite orbits if $\mathcal{O}_{\Gamma} \subset \mathcal{F}$. For any subset $\Delta \subseteq \Gamma$, let $\langle \Delta \rangle$ be the subgroup of Γ generated by Δ .¹² Say Γ has locally finite orbits if, for any finite $\Delta \subseteq \Gamma$, the subgroup $\langle \Delta \rangle$ has finite orbits.¹³

Example 11. (a) Let \mathcal{P} be any partition of \mathcal{I} into disjoint finite subsets —i.e. $\mathcal{P} \subset \mathcal{F}$ and $\mathcal{I} = \bigsqcup_{\mathcal{P} \in \mathcal{P}} \mathcal{P}$. Let $\Gamma := \{\pi \in \Pi ; \pi(\mathcal{P}) = \mathcal{P} \text{ for all } \mathcal{P} \in \mathcal{P}\}$; then $\mathcal{O}_{\Gamma} = \mathcal{P}$, so Γ has finite orbits.

(b) Π_{fin} does not have finite orbits, but it does have locally finite orbits.

(c) Let $\mathcal{I} = \mathbb{N}$. Say a permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$ is fixed step if there is some $T_{\pi} \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, we have $\pi[1 \dots N T_{\pi}] = [1 \dots N T_{\pi}]$. Let Π_{fs} be the group of all fixed step permutations.¹⁴ Then Π_{fs} does not have finite orbits, but it does have locally finite orbits. To see this, let $\Delta = \{\delta_1, \dots, \delta_K\} \subset \Pi_{\text{fs}}$, be any finite set. Let T be the lowest common multiple of $T_{\delta_1}, \dots, T_{\delta_K}$. Then for any $N \in \mathbb{N}$, we have $\delta_k[1 \dots N T] = [1 \dots N T]$ for all $k \in [1 \dots K]$; thus, $\delta[1 \dots N T] = [1 \dots N T]$ for all $\delta \in \langle \Delta \rangle$. Now, for any $n \in \mathbb{N}$, find $N \in \mathbb{N}$ with $N T \geq n$. Then $\mathcal{O}_{\langle \Delta \rangle}(n) \subseteq [1 \dots N T]$, so $\mathcal{O}_{\langle \Delta \rangle}(n)$ is finite, as desired. \diamond

¹¹For purely cosmetic reasons, one might want $* \mu[\mathcal{I}] = 1$, so that $* \mu$ seems more like a classical probability measure. This can be achieved by embedding $* \mathbb{Z}$ in $* \mathbb{Q}$, and replacing $* \mu$ with the $* \mathbb{Q}$ -valued measure $* \tilde{\mu}$ defined by $* \tilde{\mu}(\mathcal{J}) := * \mu(\mathcal{J}) / * \mu(\mathcal{I})$. One can then define the integral of an \mathcal{R} -valued function relative to $* \tilde{\mu}$ by first embedding \mathcal{R} into a lexical vector space via Hahn’s embedding theorem.

¹²That is: $\langle \Delta \rangle := \{\delta_1^{n_1} \cdot \delta_2^{n_2} \dots \delta_k^{n_k} ; k \in \mathbb{N}, \delta_1, \dots, \delta_k \in \Delta, \text{ and } n_1, \dots, n_k \in \mathbb{Z}\}$.

¹³Basu and Mitra [8] show that a permutation group $\Gamma \subset \Pi$ can be the symmetry group of some Paretian social welfare relation on $\mathbb{R}^{\mathbb{N}}$ if and only if each single element of Γ has finite orbits. The condition of locally finite orbits is similar, but somewhat more restrictive.

¹⁴ Π_{fs} -invariant social welfare relations on $\mathbb{R}^{\mathbb{N}}$ have been considered by Fleurbaey and Michel [29, §4.2] and Basu and Mitra [8, §5].

Fix a subgroup $\Gamma \subset \Pi$ with locally finite orbits. For any finite subsets $\mathcal{J} \subset \mathcal{I}$ and $\Delta \subseteq \Gamma$, let $\mathcal{F}_\Delta(\mathcal{J}) := \{\mathcal{F} \in \mathcal{F}; \mathcal{J} \subseteq \mathcal{F} \text{ and } \delta(\mathcal{F}) = \mathcal{F} \text{ for all } \delta \in \Delta\}$. Then define $\mathfrak{F}_\Gamma := \{\mathcal{E} \subseteq \mathcal{F}; \mathcal{F}_\Delta(\mathcal{J}) \subseteq \mathcal{E} \text{ for some finite } \mathcal{J} \subset \mathcal{I} \text{ and finite } \Delta \subseteq \Gamma\}$.

Lemma 12 (a) *There exists a free ultrafilter $\mathfrak{U}\mathfrak{F}$ with $\mathfrak{F}_\Gamma \subseteq \mathfrak{U}\mathfrak{F}$.*

(b) $\Pi_{\text{fin}} \cup \Gamma \subseteq \Pi_{\mathfrak{U}\mathfrak{F}}$.

Example 13. Let $\mathcal{I} = \mathbb{R}$. Any element of $r \in \mathbb{R}$ has a unique *binary expansion* $r = \sum_{z \in \mathbb{Z}} r_z 2^z$ for some sequence $\{r_z\}_{z \in \mathbb{Z}}$ taking values in $\{0, 1\}$, such that: (i) there exists some $N \in \mathbb{N}$ such that $r_z = 0$ for all $z > N$; and (ii) for any $M \in \mathbb{N}$, there exists some $z < -M$ with $r_z = 0$.¹⁵ For all $n \in \mathbb{Z}$, define $\pi_n : \mathbb{R} \rightarrow \mathbb{R}$ by $\pi_n(r) := (1 - r_n)2^n + \sum_{z \in \mathbb{Z} \setminus \{n\}} r_z 2^z$ (i.e. toggle the n th binary digit of r). Let Γ be the group generated by $\{\pi_n\}_{n \in \mathbb{Z}}$; then Γ has locally finite orbits (because any finite subset of Γ can only act upon a finite set of digits).

Let $\mathfrak{U}\mathfrak{F}$ be as in Lemma 12; then $\Gamma \subseteq \Pi_{\mathfrak{U}\mathfrak{F}}$. Let ${}^*\mu$ be the ${}^*\mathbb{Z}$ -valued measure on \mathbb{R} defined by $\mathfrak{U}\mathfrak{F}$ in Example 8. Then for any $\mathcal{J} \subseteq \mathbb{R}$ and $n \in \mathbb{Z}$, we have ${}^*\mu[\pi_n(\mathcal{J})] = {}^*\mu[\mathcal{J}]$. This forces ${}^*\mu$ to be very similar to the Lebesgue measure. For example, let ${}^*\epsilon := {}^*\mu[0, 1)$. It is easy to check that ${}^*\mu\left[\frac{n}{2^k}, \frac{m}{2^k}\right) = \frac{(m-n)}{2^k} {}^*\epsilon$ for any $n, m, k \in \mathbb{Z}$ with $n < m$. An interval of this kind is called a *dyadic interval*. A *dyadic subset* is a finite disjoint union of dyadic intervals. If \mathcal{J} is any dyadic subset, with Lebesgue measure $\lambda(\mathcal{J})$, it follows that ${}^*\mu(\mathcal{J}) = \lambda(\mathcal{J}) \cdot {}^*\epsilon$. \diamond

From now on, define ${}^*\mathcal{R}$ and $(\overset{*}{\succeq}_u)$ using the ultrafilter $\mathfrak{U}\mathfrak{F}$ from Lemma 12. Lemma 12(b) implies that any $\Pi_{\mathfrak{U}\mathfrak{F}}$ -invariant preorder is automatically Π_{fin} -invariant. Thus, Lemma 10(b) implies that $(\overset{*}{\succeq}_u)$ is Π_{fin} -invariant.

If (\succeq) is any finitary preorder on $\mathcal{X}^{\mathcal{I}}$, then the *finitary part* of (\succeq) is the strictly finitary preorder $(\overset{\text{fin}}{\succeq})$ defined by $(\mathbf{x} \overset{\text{fin}}{\succeq} \mathbf{y}) \iff (\mathbf{x} \succeq \mathbf{y} \text{ and } d(\mathbf{x}, \mathbf{y}) < \infty)$.

Proposition 14 *If $\mathfrak{U}\mathfrak{F}$ is defined as in Lemma 12(a), then the finitary part of $(\overset{*}{\succeq}_u)$ is the additive preorder $(\overset{\text{fin}}{\succeq}_u)$.*

For any $\mathbf{x}, \mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and $\mathcal{J} \subseteq \mathcal{I}$, let $\mathbf{x}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}}$ denote the element $\mathbf{w} \in \mathcal{X}^{\mathcal{I}}$ defined by $w_j := x_j$ for all $j \in \mathcal{J}$ and $w_i := z_i$ for all $i \in \mathcal{I} \setminus \mathcal{J}$.

Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, and let $\mathcal{G} \in \mathfrak{U}\mathfrak{F}$. Write “ $\mathbf{x} \overset{\mathcal{G}}{\succeq} \mathbf{y}$ ” if, for all $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and all $\mathcal{J} \in \mathcal{G}$, we have $\mathbf{x}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}} \succeq \mathbf{y}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}}$. (Heuristically, this means $\mathbf{x}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}} \succeq \mathbf{y}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}}$ for ‘almost all’ finite subsets $\mathcal{J} \subset \mathcal{I}$.) Write “ $\mathbf{x} \overset{\mathcal{G}}{\succ} \mathbf{y}$ ” if, for all $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and all $\mathcal{J} \in \mathcal{G}$, we have $\mathbf{x}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}} \succ \mathbf{y}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}}$.¹⁶ The preorder (\succeq) is *$\mathfrak{U}\mathfrak{F}$ -continuous* if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$:

(C1) if $\mathbf{x} \overset{\mathcal{G}}{\succeq} \mathbf{y}$ for some $\mathcal{G} \in \mathfrak{U}\mathfrak{F}$, then $\mathbf{x} \succeq \mathbf{y}$.

(C2) if $\mathbf{x} \overset{\mathcal{G}}{\succ} \mathbf{y}$ for some $\mathcal{G} \in \mathfrak{U}\mathfrak{F}$, then $\mathbf{x} \succ \mathbf{y}$.

¹⁵Condition (i) is because r is finite. Condition (ii) ensures uniqueness by excluding binary expansions ending in an infinite sequence of 1’s.

¹⁶If (\succeq) is separable, then we could equivalently define the relations $(\overset{\mathcal{G}}{\succeq})$ and $(\overset{\mathcal{G}}{\succ})$ by requiring these conditions to hold for *some* $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, rather than for *all* $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$.

Lemma 15 $(\overset{*}{\succeq}_u)$ is $\mathcal{U}\mathfrak{F}$ -continuous. In fact, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have

$$(a) \quad (\mathbf{x} \overset{*}{\succeq}_u \mathbf{y}) \iff (\mathbf{x} \overset{*}{\succeq}_{u, \mathcal{G}} \mathbf{y} \text{ for some } \mathcal{G} \in \mathcal{U}\mathfrak{F}).$$

$$(b) \quad (\mathbf{x} \overset{*}{\succ}_u \mathbf{y}) \iff (\mathbf{x} \overset{*}{\succ}_{u, \mathcal{G}} \mathbf{y} \text{ for some } \mathcal{G} \in \mathcal{U}\mathfrak{F}).$$

Lemma 16 Let (\succeq) be a preorder on $\mathcal{X}^{\mathcal{I}}$, and let $u : \mathcal{X} \rightarrow \mathcal{R}$. Then

$$\left((\overset{\text{fin}}{\succeq}) = (\overset{\text{fin}}{\succeq}_u), \text{ and } (\succeq) \text{ is } \mathcal{U}\mathfrak{F}\text{-continuous} \right) \iff \left((\succeq) = (\overset{*}{\succeq}_u) \right).$$

The hypothesis of $\mathcal{U}\mathfrak{F}$ -continuity seems to be necessary for the conclusion of Theorem 2. To see this, let ${}^*\mu$ be the ${}^*\mathbb{N}$ -valued measure on \mathcal{I} from Example 8, and define ${}^*I := {}^*\mu(\mathcal{I})$ (an infinite hypernatural number). Let $\mathcal{R} := \mathbb{R}$, let $\text{st} : {}^*\mathbb{R} \rightarrow \mathbb{R} \sqcup \{\pm\infty\}$ be the standardization homomorphism,¹⁷ and define the relation (\succeq) on $\mathcal{X}^{\mathcal{I}}$ by:

$$(\mathbf{x} \succeq \mathbf{y}) \iff \left(\text{st} \left(\frac{1}{{}^*I} \sum_{i \in \mathcal{I}} u(x_i) \right) \geq \text{st} \left(\frac{1}{{}^*I} \sum_{i \in \mathcal{I}} u(y_i) \right) \right).$$

Then (\succeq) is complete, separable, and $\Pi_{\mathcal{U}\mathfrak{F}}$ -invariant. However, the finitary part of (\succeq) is trivial; thus, there is no nontrivial utility function u such that $(\succeq) = (\overset{*}{\succeq}_u)$. Indeed, (\succeq) is much coarser than any hyperadditive preorder; for example, if \mathcal{I} is uncountable, then (\succeq) is indifferent between \mathbf{x} and \mathbf{y} whenever $\mathcal{I}(\mathbf{x}, \mathbf{y})$ is countable.

4 About the axioms

Π_{fin} -invariance does not require (\succeq) to be invariant under arbitrary permutations of \mathcal{I} ; in this sense, it lacks the full ethical force of the standard ‘anonymity’ axiom of social choice theory. Fortunately, $(\overset{*}{\succeq}_u)$ is invariant under a somewhat larger group $\Pi_{\mathcal{U}\mathfrak{F}}$ of permutations, which includes some (but not all) non-finitary ones (Lemma 10(b)). Indeed, given any permutation group $\Gamma \subset \Pi$ with locally finite orbits, one can construct $\mathcal{U}\mathfrak{F}$ so that $\Gamma \subset \Pi_{\mathcal{U}\mathfrak{F}}$ (Lemma 12). Of course, $\Pi_{\mathcal{U}\mathfrak{F}}$ is still only a small subgroup of the group of all permutations of \mathcal{I} . However, it is well-known that (\succeq) cannot be invariant under *all* permutations of \mathcal{I} and also satisfy the Pareto/dominance axiom (5).¹⁸

To understand $\mathcal{U}\mathfrak{F}$ -continuity, suppose (\succeq) is separable. Then for each subset $\mathcal{J} \subseteq \mathcal{I}$, there exists a preorder $(\overset{\text{fin}}{\succeq}_{\mathcal{J}})$ on $\mathcal{X}^{\mathcal{J}}$ such that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}} \succeq \mathbf{y}_{\mathcal{J}} \mathbf{z}_{\mathcal{I} \setminus \mathcal{J}}$ if and only if $\mathbf{x}_{\mathcal{J}} \overset{\text{fin}}{\succeq}_{\mathcal{J}} \mathbf{y}_{\mathcal{J}}$. Heuristically, if $\mathcal{G} \in \mathcal{U}\mathfrak{F}$, and a statement holds for all $\mathcal{J} \in \mathcal{G}$, then this statement holds for ‘almost all’ finite subsets of \mathcal{I} . Thus, axiom (C1) says: “if $\mathbf{x}_{\mathcal{J}} \overset{\text{fin}}{\succeq}_{\mathcal{J}} \mathbf{y}_{\mathcal{J}}$ for almost all finite subsets $\mathcal{J} \subset \mathcal{I}$, then $\mathbf{x} \succeq \mathbf{y}$ ”. Axiom (C2) has a similar interpretation, but for strict preferences.

¹⁷Standardization maps all infinite hyperreals to $\pm\infty$, and strips the infinitesimal part away from all finite hyperreals, leaving only the real part.

¹⁸Lauwers [41], Basu and Mitra [7, 8, 9], Fleurbaey and Michel [29, Thm. 1] and Sakai [51, Thm.2] have analyzed this Pareto/anonymity conflict in greater detail.

Axiom (C1) is very similar to the axiom of *Time-Invariant Preference Continuity* (IPC) proposed by Asheim *et al.* [3]. The difference is that IPC only considers sets of the form $\mathcal{G}_{\mathcal{J}} := \{\mathcal{F} \in \mathcal{F}; \mathcal{J} \subseteq \mathcal{F}\}$, for some finite $\mathcal{J} \subset \mathcal{I}$; in contrast, (C1) allows \mathcal{G} to be any element of $\mathcal{U}\mathfrak{F}$. Axiom (C1) is also similar to Fleurbaey and Michel’s [29] ‘Limit Ranking’ axiom, or part (a) of Basu and Mitra’s [10, Axiom 4] ‘Strong consistency’. Axiom (C2) is similar to Asheim and Tungodden’s [4, WPC] ‘Weak Preference Consistency’, or part (b) of Basu and Mitra’s [10, Axiom 5] ‘Weak consistency’. The difference is that these three papers suppose $\mathcal{I} = \mathbb{N}$ and specify a particular choice of \mathcal{G} (namely: $\mathcal{G} := \{[1 \dots T]; T \in \mathbb{N}\}$), whereas $\mathcal{U}\mathfrak{F}$ -continuity allows \mathcal{G} to be any element of $\mathcal{U}\mathfrak{F}$.

Separability imposes a mild restriction on attitudes towards (i) intertemporal volatility, (ii) risk, and/or (iii) interpersonal inequality. If $\mathcal{I} = \mathcal{J} \sqcup \mathcal{K}$, then separability means that preferences over $\mathbf{x}_{\mathcal{K}}$ versus $\mathbf{y}_{\mathcal{K}}$ must be independent of whether the outcomes encoded in $\mathbf{x}_{\mathcal{K}}$ and/or $\mathbf{y}_{\mathcal{K}}$ are on average much better or much worse than the outcomes encoded in $\mathbf{x}_{\mathcal{J}}$ and/or $\mathbf{y}_{\mathcal{J}}$. For example, improving x_i to y_i has the same social value, whether $i \in \mathcal{I}$ is currently the *least* happy person, time period, or state of nature in \mathbf{x} , or already the *most* happy person, time period, or state of nature in \mathbf{x} .¹⁹ This excludes ‘rank-dependent expected utility’ models of risky choice, and excludes ‘rank-weighted utilitarian’ social welfare orders such as the ‘generalized Gini’ family.

We can still encode a considerable amount of volatility/risk/inequality aversion into (\succeq_u) and $(\overset{*}{\succeq}_u)$ by defining $u := f \circ w$, where $w : \mathcal{X} \rightarrow \mathbb{R}$ measures the ‘true’ welfare level of the outcomes in \mathcal{X} on some cardinal scale, and $f : \mathbb{R} \rightarrow \mathcal{R}$ is a concave increasing function (see Example 4(v)). However, this merely strengthens the grip of the Repugnant Conclusion: the more concave f is, the easier it becomes for a large population of uniformly miserable people to dominate a smaller population of happy people. Also, in a model like Example 4(iv), u acts like a vNM utility function describing each individual’s lottery preferences. Thus, the risk-preferences of the social preorders (\succeq_u) and $(\overset{*}{\succeq}_u)$ must exactly match the risk-preferences of the individuals.

5 Related literature

There are at least three ways to obtain an additive utility representation for a separable preference order on a Cartesian product. The ‘subjective expected utility’ approach, originating with Savage [53], begins with a separable, nonatomic preference relation on $\mathcal{X}^{\mathcal{I}}$, and represents it using a utility function on \mathcal{X} and a subjective probability measure on \mathcal{I} (see Example 4(ii) above). The ‘topological’ approach, originating with Debreu [23] and Gorman [30], concerns a continuous preference relation on a product of (connected) topological spaces. The ‘algebraic’ approach, exemplified by Krantz *et al.* [40], concerns a complete order relation (sometimes called an ‘additive conjoint measurement’) defined on a product of abstract, often finite, and generally nonidentical sets, and supplements separability with various ‘higher-order cancellation’ and ‘solvability’ axioms. Fishburn [26], Narens [46] and Wakker [59] are good references for these theories.

¹⁹Of course, individuals can still derive (dis)utility from memory of the past, anticipation of the future, altruism/envy towards other people, or the contingency of fate, as long as the relevant cognitive states are explicitly encoded in \mathcal{X} .

Since it involves no topological assumptions, Theorem 1 is closest to the algebraic tradition, but with three notable differences. First, aside from separability, Theorem 1 invokes only one simple hypothesis: permutation invariance (which is inapplicable to many other models in measurement/utility theory). Second, most of the existing literature invokes an Archimedean condition to ensure a real-valued representation (with the exceptions discussed in §5.1 below). Third, almost all of the existing literature in both the topological and algebraic traditions concerns *finite* Cartesian products, whereas Theorem 1 is about infinite Cartesian products. An important exception is Wakker [58], who proves a theorem equivalent to Corollary 5.²⁰ Indeed, instead of being a problem, the infinite cardinality of \mathcal{I} plays an essential role in the construction of the utility representation, both for Wakker [58] and for our proof of Theorem 1.

Theorem 2 is not purely ‘algebraic’, since it does invoke a weak continuity condition. But this condition has nothing to do with any topology on \mathcal{X} , and is introduced only to deal with the infinite cardinality of \mathcal{I} .

5.1 Non-Archimedean utility and probability

Let $(\mathcal{N}, >)$ be a totally ordered set. For any $\mathbf{r} \in \mathbb{R}^{\mathcal{N}}$, let $\mathcal{N}(\mathbf{r}) := \{n \in \mathcal{N}; r_n \neq 0\}$. Let $\mathcal{L} := \{\mathbf{r} \in \mathbb{R}^{\mathcal{N}}; \mathcal{N}(\mathbf{r}) \text{ is well-ordered by } >\}$. Then \mathcal{L} is a linear subspace of $\mathbb{R}^{\mathcal{N}}$. (If \mathcal{N} is well-ordered—in particular, if \mathcal{N} is finite—then $\mathcal{L} = \mathbb{R}^{\mathcal{N}}$.) Define the *lexicographical order* \gg on \mathcal{L} by setting $\mathbf{r} \gg \mathbf{s}$ if $r_n > s_n$, where $n := \min[\mathcal{N}(\mathbf{r} - \mathbf{s})]$. Then (\mathcal{L}, \gg) is a linearly ordered vector space, which is Archimedean if and only if $\mathcal{N} = \{1\}$. Call \mathcal{L} a *lexical vector space*, and call a utility function $u : \mathcal{X} \rightarrow \mathcal{L}$ a *lexicographical utility function*.

Hahn’s Embedding Theorem says that any linearly ordered abelian group can be isomorphically embedded as an ordered subgroup of a lexical vector space; see Gravett [31] for details. Similarly, Hausner and Wendel [37] showed that any linearly ordered vector space admits a linear embedding as an ordered subspace of a lexical vector space. Thus, any non-Archimedean utility function can be reduced to a lexicographical utility function.

Hausner [36] was the first to study lexicographical von Neumann-Morgenstern (vNM) utility functions. Chipman [21, Thm. 3.1] showed that any totally ordered set (\mathcal{X}, \succeq) could be represented by an ordinal lexicographical utility function.²¹ He also [21, §3.5] discussed how to construct lexicographical cardinal utility functions, compatible with a certain measure of ‘preference intensity’ on (\mathcal{X}, \succeq) . Since then, there has been a great deal of interest in lexicographical utility functions; Fishburn ([25, §5-§6] and [26, Chapt.4]) gives two good surveys of the earlier literature, while Blume *et al.* [14] and Halpern [33, §1] survey more recent developments. Fishburn and Lavalley [27] and Fishburn [24, §5] summarize a series of papers in which these two authors developed a theory of finite-

²⁰Wakker [58] builds on Camacho’s [17, 18, 19, 20] earlier theory of additive preferences over the space of arbitrary-length finite sequences in \mathcal{X} . More recently, Wakker and Zank [60] have developed another additive representation for infinite Cartesian powers, using using measure theory. Meanwhile, Streufert [56] has analyzed the separability structure of partially separable (but not necessarily additive) preferences on infinite Cartesian products.

²¹See also Chipman [22]. In fact, this result had earlier been proved independently in papers by Sierpiński, Cuesta, and Mendelson; see [25, §5] for details. Starting from Chipman’s work, Herden and Mehta [38] have developed *continuous* lexicographical ordinal utility functions.

dimensional lexicographical vNM utilities and ‘matrix-valued’ probabilities.

Much of this literature either combines lexicographical probabilities with real-valued utilities, or combines lexicographical utilities with real-valued (or matrix-valued) probabilities. The reason is that there is no way to ‘multiply’ two elements of a lexical vector space. (This also makes it difficult to define ‘conditional probability’ for lexicographical probabilities.) In contrast, it *is* possible to multiply elements of the field ${}^*\mathbb{R}$, so one can easily combine ${}^*\mathbb{R}$ -valued utility with ${}^*\mathbb{R}$ -valued probability. Richter [48] was the first to suggest the use of ${}^*\mathbb{R}$ -valued vNM utilities and/or subjective probabilities. Skala [54, 55] and Narens ([45] and [46, Chapt.4-6]) developed this approach in more detail. Blume *et al.* [15], Hammond [34] and Halpern [33] study both lexicographical and ${}^*\mathbb{R}$ -valued probabilities, and conclude that the latter model subsumes the former.

Blume *et al.* [16] apply their theory of lexicographical and ${}^*\mathbb{R}$ -valued probabilities to explicate equilibrium refinements in game theory. This idea has since been explored by several authors, most recently Halpern [32]. Unfortunately, aside from Fleurbaey and Michel [29, Thm.5], there seems to have been no application of ${}^*\mathbb{R}$ -valued utilities to infinite-horizon intertemporal choice.

5.2 Intertemporal and intergenerational choice

Let $\mathcal{I} = \mathbb{N}$, and for all $t \in \mathbb{N}$, interpret x_t and $u(x_t)$ as, respectively, the social outcome and social welfare at time t . Then Theorems 1 and 2 are comparable to several previous extensions of utilitarianism to infinite-horizon intertemporal choice. For example, let ℓ^∞ be the set of bounded, real-valued sequences indexed by \mathbb{N} . Any ultrafilter $\mathcal{G} \subset 2^\mathbb{N}$ defines a linear map $\lim_{\mathcal{G}} : \ell^\infty \rightarrow \mathbb{R}$. Given any ‘utility stream’ $(u_t)_{t=1}^\infty \in \ell^\infty$, one can then compute the $\lim_{\mathcal{G}}$ of the sequence of time averages $\{\frac{1}{T} \sum_{t=1}^T u_t\}_{T=1}^\infty$ as $T \rightarrow \infty$, to obtain a sort of ‘Cesáro average’ (dependent on \mathcal{G}). One can then linearly combine the Cesáro averages induced by many different ultrafilters; the result is a continuous, linear function $F : \ell^\infty \rightarrow \mathbb{R}$, called a *medial limit*, and can be used to define a (complete) social welfare order on ℓ^∞ . Lauwers [41, Prop.3] characterizes the family of medial limits as the unique continuous, linear, real-valued functions on ℓ^∞ which satisfy a strong anonymity condition (invariance under all ‘bounded’ permutations). However, medial limits violate the strong Pareto condition (5), because they are insensitive to a change in any finite number of coordinates. Indeed Lauwers [41, Prop.1] shows that his strong anonymity axiom is incompatible with Pareto.

Given two ‘utility streams’ $\mathbf{u}, \mathbf{v} \in \mathbb{R}^\mathbb{N}$, define $\mathbf{u} \succeq \mathbf{v}$ if there is some $T \in \mathbb{N}$ such that $\sum_{t=1}^T u_t \geq \sum_{t=1}^T v_t$ while $\mathbf{u}_{[T..\infty]}$ Pareto dominates $\mathbf{v}_{[T..\infty]}$. This (incomplete) social welfare order is characterized by Basu and Mitra [10, Thm.1]. Banerjee [6, §4] characterizes an extension of the Basu-Mitra relation where \mathbf{u} and/or \mathbf{v} can be transformed by some admissible permutation prior to comparison. Unlike Lauwer’s medial limit order, the Basu-Mitra-Banerjee preorder satisfies the strong Pareto condition (5); however, it is quite incomplete. Furthermore, both Basu and Mitra [10] and Banerjee [6] use an axiom of ‘partial translation scale invariance’ to make utility cardinal and interpersonally comparable. (Likewise, Lauwer’s [41] characterization invokes linearity). In contrast, Theorems 1 and 2 do not impose any scale-invariance condition on utility —indeed, the utility function u is not even a primitive of the model, but emerges as a consequence of the separability axiom.

The definition of $(\overset{*}{\underset{u}{\succ}})$ is also reminiscent of the ‘catching-up’ preorder $(\underset{u}{\succ}^c)$ and ‘overtaking’ preorder $(\underset{u}{\succ}^o)$ originally proposed by Atsumi [5] and von Weizsacker [57]. Formally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \underset{u}{\succ}^c \mathbf{y}$ iff there exists $T_0 \in \mathbb{N}$ such that $\sum_{t=1}^T u(x_t) \geq \sum_{t=1}^T u(y_t)$ for all $T \geq T_0$. Likewise, $\mathbf{x} \underset{u}{\approx}^o \mathbf{y}$ iff there exists $T_0 \in \mathbb{N}$ such that $\sum_{t=1}^T u(x_t) = \sum_{t=1}^T u(y_t)$ for all $T \geq T_0$; meanwhile $\mathbf{x} \underset{u}{\succ}^o \mathbf{y}$ iff there exists $T_0 \in \mathbb{N}$ such that $\sum_{t=1}^T u(x_t) > \sum_{t=1}^T u(y_t)$ for all $T \geq T_0$. Asheim and Tungodden [4, Propositions 4 and 5] and Basu and Mitra [10, Theorems 2 and 3] provide axiomatic characterizations. Sakai [52] proposes a family of similar orderings called *future agreement extensions*.

Clearly, $(\mathbf{x} \underset{u}{\approx}^o \mathbf{y}) \implies (\mathbf{x} \underset{u}{\succ}^c \mathbf{y})$. However, as observed by Banerjee [6, §5], both $(\underset{u}{\succ}^c)$ and $(\underset{u}{\approx}^o)$ are rather incomplete. Furthermore, their definitions clearly depend on the ordering of \mathbb{N} , which is appropriate for intertemporal choice, but somewhat dubious for interpretations (ii) and (iii). Lauwers and Vallentyne [43, Thm.4] generalize $(\underset{u}{\succ}^c)$ to any countable, unordered set, by defining the *weak catching-up* preorder: $\mathbf{x} \underset{u}{\succ}^w \mathbf{y}$ iff there exists some $\mathcal{F} \in \mathcal{F}$ such that $\sum_{j \in \mathcal{J}} u(x_j) \geq \sum_{j \in \mathcal{J}} u(y_j)$ for all $\mathcal{J} \in \mathcal{F}$ with $\mathcal{F} \subseteq \mathcal{J}$. Asheim *et al.* [3] propose a similar relation, the *Generalized time-invariant overtaking criterion*, generated by any ‘proliferating’ sequence of finite-dimensional SWOs. However $(\underset{u}{\succ}^w)$ is even more incomplete than $(\underset{u}{\succ}^c)$. Clearly, $(\mathbf{x} \underset{u}{\succ}^w \mathbf{y}) \implies (\mathbf{x} \overset{*}{\underset{u}{\succ}} \mathbf{y})$. But the ultrafilter axiom (UF) ensures that $(\overset{*}{\underset{u}{\succ}})$ is complete.

5.3 Ultrafilters and aggregation

The close relationship between preference aggregation and ultrafilters was first noted by Kirman and Sondermann [39], and thoroughly analyzed by Lauwers and van Liedekerke [44]. In their analysis, the elements of the ultrafilter represent decisive coalitions of voters; thus, Kirman and Sondermann interpret a free ultrafilter as defining an ‘invisible dictator’. However, Fleurbaey and Michel [29, p.792] observe that this interpretation is inappropriate for $(\overset{*}{\underset{u}{\succ}})$, because the elements of $\mathcal{U}\mathfrak{F}$ now do not represent decisive subsets of \mathcal{I} , but instead represent ‘large’ subsets of \mathcal{F} (operationalizing some notion of summation over a ‘generic finite subset’ of \mathcal{I}).

A more serious objection to $(\overset{*}{\underset{u}{\succ}})$ is that, since it is defined through an ultrafilter, it is not explicitly constructable within the Zermelo-Fraenkel (ZF) axioms. But this is unavoidable: Zame [61] and Lauwers [42] have both shown that *any* ‘reasonable’ infinite-horizon intertemporal preference order is nonconstructable in ZF. This is closely related to another problem. Let $\Gamma \subset \Pi$ be a permutation subgroup with locally finite orbits. Say that a free ultrafilter $\mathcal{U}\mathfrak{F}$ is Γ -*admissible* if $\mathfrak{F}_\Gamma \subseteq \mathcal{U}\mathfrak{F}$, as in Lemma 12(a). There are an uncountable number of distinct Γ -admissible ultrafilters on \mathcal{F} , and for any such ultrafilter \mathcal{U} , we can define an ultrapower ${}^*\mathcal{R}^{(\mathcal{U})}$ and a corresponding version $(\overset{*}{\underset{u}{\succ}}^{\mathcal{U}})$ of the hyperadditive ordering on $\mathcal{X}^{\mathcal{I}}$. If \mathcal{U} and \mathcal{V} are two such ultrafilters, then $(\overset{*}{\underset{u}{\succ}}^{\mathcal{U}})$ and $(\overset{*}{\underset{u}{\succ}}^{\mathcal{V}})$ will be different orderings, and will produce opposite rankings for certain pairs of elements in $\mathcal{X}^{\mathcal{I}}$. This makes the ordering between these elements seem somewhat arbitrary. But there is a partial resolution of this ambiguity, which is sufficient for most practical purposes.

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we define $\mathbf{x} \underset{u}{\succ}^\Gamma \mathbf{y}$ iff there exists some finite $\mathcal{E} \subset \mathcal{I}$ and finite $\Delta \subseteq \Gamma$ such that $\sum_{j \in \mathcal{J}} u(x_j) \geq \sum_{j \in \mathcal{J}} u(y_j)$ for all $\mathcal{J} \in \mathcal{F}_\Delta(\mathcal{E})$. Thus, $(\underset{u}{\succ}^\Gamma) \subseteq (\overset{*}{\underset{u}{\succ}})$, with equality if Γ is trivial. If $\Gamma \subseteq \Theta \subset \Pi$, then $(\underset{u}{\succ}^\Gamma) \subseteq (\underset{u}{\succ}^\Theta)$. Thus, the larger Γ is, the

greater the scope of the next result:

Proposition 17 *Let \mathcal{R} be a linearly ordered abelian group and let $u : \mathcal{X} \rightarrow \mathcal{R}$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \succeq_u^{\Gamma} \mathbf{y}$ if and only if $\mathbf{x} \stackrel{*}{\succeq}_u^{\mathfrak{U}} \mathbf{y}$ for every Γ -admissible ultrafilter \mathfrak{U} on \mathcal{F} .*

It is often ZF-decidable whether $\mathbf{x} \succeq_u^{\Gamma} \mathbf{y}$, and in this case, we can immediately deduce that $\mathbf{x} \stackrel{*}{\succeq}_u^{\mathfrak{U}} \mathbf{y}$ without knowing anything about the structure of \mathfrak{U} . On the other hand, if neither $\mathbf{x} \succeq_u^{\Gamma} \mathbf{y}$ nor $\mathbf{x} \preceq_u^{\Gamma} \mathbf{y}$, then the difference between \mathbf{x} and \mathbf{y} is so subtle that perhaps we can treat them as ‘virtually indifferent’ for practical purposes.

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Appendix: Proofs

Proof of Theorem 1. “ \Leftarrow ” It is easy to check that the additive preorder defined by (2) is Π_{fin} -invariant and separable.

“ \Rightarrow ” Fix some $o \in \mathcal{X}$, and let $(\mathcal{A}, +)$ be the free abelian group generated by $\mathcal{X} \setminus \{o\}$. That is, \mathcal{A} consists of all formal \mathbb{Z} -linear combinations of the form “ $J_1x_1 + J_2x_2 + \dots + J_Nx_N$ ” where $N \in \mathbb{N}$, $J_1, \dots, J_N \in \mathbb{Z} \setminus \{0\}$, and $x_1, \dots, x_N \in \mathcal{X} \setminus \{o\}$ are distinct.

Let $\mathcal{B}' \subset \mathcal{A}$ be the set of all elements where $J_1, \dots, J_N > 0$. For any such $b \in \mathcal{B}'$, we define $\mathbf{w}^b \in \mathcal{X}^{\mathcal{I}}$ as follows. Let $\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_N$ be disjoint subsets of \mathcal{I} , with $|\mathcal{J}_n| = J_n$ for all $n \in [1 \dots N]$. For all $n \in [1 \dots N]$ and all $j \in \mathcal{J}_n$, let $w_j^b := x_n$. Meanwhile, for all $i \in \mathcal{I} \setminus \mathcal{J}_1 \sqcup \dots \sqcup \mathcal{J}_N$, define $w_i^b := o$. (Heuristic: if we regard the elements of \mathcal{X} as ‘goods’ and ‘bads’, then \mathbf{w}^b represents a ‘bundle’ containing J_n units of x_n for each $n \in [1 \dots N]$.) Because (\succeq) is Π_{fin} -invariant, it doesn’t matter how we choose the sets $\mathcal{J}_1, \dots, \mathcal{J}_N$; if \mathbf{w}^b and $\tilde{\mathbf{w}}^b$ are two different elements of $\mathcal{X}^{\mathcal{I}}$ built using the above recipe, then we automatically have $\mathbf{w}^b \approx \tilde{\mathbf{w}}^b$.

Finally, if 0 denotes the identity element of \mathcal{A} , then let $\mathcal{B} := \{0\} \sqcup \mathcal{B}'$, and define \mathbf{w}^0 by setting $w_i^0 := o$ for all $i \in \mathcal{I}$. For any $b, b' \in \mathcal{B}$, if $b = J_1x_1 + \dots + J_Nx_N$ and $b' = J'_1x'_1 + \dots + J'_Mx'_M$, then say b and b' are *disjoint* if the sets $\{x_1, \dots, x_N\}$ and $\{x'_1, \dots, x'_M\}$ are disjoint.

Claim 1: *Let $a \in \mathcal{A}$.*

(a) *There exist unique disjoint $a_+, a_- \in \mathcal{B}$ such that $a = a_+ - a_-$.*

(b) Let $b, c \in \mathcal{B}$ be any other elements such that $a = b - c$. Then $(\mathbf{w}^b \succeq \mathbf{w}^c) \iff (\mathbf{w}^{a+} \succeq \mathbf{w}^{a-})$.

Proof. (a) If $a = 0$, then let $0_+ := 0$ and $0_- := 0$. Then $0_+, 0_- \in \mathcal{B}$ are disjoint, and $0 = 0_+ - 0_-$.

Now suppose $a \neq 0$. Let $a = \sum_{w \in \mathcal{W}} A_w w$, where $\mathcal{W} \subseteq \mathcal{X}$ is a finite subset and $A_w \in \mathbb{Z} \setminus \{0\}$ for all $w \in \mathcal{W}$. Then $\mathcal{W} := \mathcal{W}_- \sqcup \mathcal{W}_+$, where $\mathcal{W}_- := \{w \in \mathcal{W}; A_w < 0\}$ and $\mathcal{W}_+ := \{w \in \mathcal{W}; A_w > 0\}$. Let $a_+ := \sum_{w \in \mathcal{W}_+} A_w w$ and $a_- := \sum_{w \in \mathcal{W}_-} (-A_w)w$. (If $\mathcal{W}_+ = \emptyset$, then $a_+ := 0$. If $\mathcal{W}_- = \emptyset$, then $a_- := 0$.) Then $a_+, a_- \in \mathcal{B}$ are disjoint and $a = a_+ - a_-$.

(b) Suppose $b := \sum_{y \in \mathcal{Y}} B_y y$ and $c := \sum_{z \in \mathcal{Z}} C_z z$, for some finite subsets $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{X}$ and coefficients $B_y \in \mathbb{N}$ for all $y \in \mathcal{Y}$ and $C_z \in \mathbb{N}$ for all $z \in \mathcal{Z}$. If $b - c = a$, then we must have $\mathcal{W} \subseteq \mathcal{Y} \cup \mathcal{Z}$, $\mathcal{Y} \setminus \mathcal{Z} \subseteq \mathcal{W}_+$, and $\mathcal{Z} \setminus \mathcal{Y} \subseteq \mathcal{W}_-$. Furthermore:

- $B_y = A_y$ for all $y \in \mathcal{Y} \setminus \mathcal{Z}$.
- $C_z = -A_z$ for all $z \in \mathcal{Z} \setminus \mathcal{Y}$.
- $B_w - C_w = A_w$ for all $w \in \mathcal{Y} \cap \mathcal{Z} \cap \mathcal{W}$.
- $B_x = C_x$ for all $x \in (\mathcal{Y} \cap \mathcal{Z}) \setminus \mathcal{W}$.

Let $J := \max(\{B_y; y \in \mathcal{Y}\} \cup \{C_z; z \in \mathcal{Z}\})$. For all $x \in \mathcal{Y} \cup \mathcal{Z}$, let $\mathcal{J}_x \subset \mathcal{I}$ be a subset of cardinality J , and suppose all these sets are disjoint. Because (\succeq) is Π_{fin} -invariant, we can permute the coordinates of \mathbf{w}^b and/or \mathbf{w}^c each in any desired finitary way without changing their (\succeq) -ordering. Likewise, we can finitarily permute \mathbf{w}^{a+} and/or \mathbf{w}^{a-} without changing their (\succeq) -ordering. Thus, without loss of generality, we can suppose:

- \mathbf{w}^{a+} assigns the value w to exactly A_w coordinates in \mathcal{J}_w , for each $w \in \mathcal{W}_+$.
- \mathbf{w}^{a-} assigns the value w to exactly $-A_w$ coordinates in \mathcal{J}_w , for each $w \in \mathcal{W}_-$.
- \mathbf{w}^b assigns the value y to exactly B_y coordinates in \mathcal{J}_y , for each $y \in \mathcal{Y}$.
- \mathbf{w}^c assigns the value z to exactly C_z coordinates in \mathcal{J}_z , for each $z \in \mathcal{Z}$.

We describe these as the ‘active’ coordinates. In all four cases, we assign all other (‘inactive’) coordinates the value o . We can further assume that:

1. For all $y \in \mathcal{Y} \setminus \mathcal{Z}$, \mathbf{w}^{a+} and \mathbf{w}^b involve the *same* $B_y = A_y$ active coordinates.
2. For all $z \in \mathcal{Z} \setminus \mathcal{Y}$, \mathbf{w}^{a-} and \mathbf{w}^c involve the *same* $C_z = -A_z$ active coordinates.
3. For all $x \in (\mathcal{Y} \cap \mathcal{Z}) \setminus \mathcal{W}$, \mathbf{w}^b and \mathbf{w}^c involve the *same* set $\mathcal{I}_x \subseteq \mathcal{J}_x$ of $B_x = C_x$ active coordinates.
4. For all $w \in \mathcal{Y} \cap \mathcal{Z} \cap \mathcal{W}_+$, the set of active coordinates in \mathbf{w}^b is the disjoint union of the active coordinates of \mathbf{w}^{a+} and \mathbf{w}^c . In this case, let $\mathcal{I}_w \subseteq \mathcal{J}_w$ be the active coordinates of \mathbf{w}^c .
5. For all $w \in \mathcal{Y} \cap \mathcal{Z} \cap \mathcal{W}_-$, the set of active coordinates in \mathbf{w}^c is the disjoint union of the active coordinates of \mathbf{w}^{a-} and \mathbf{w}^b . In this case, let $\mathcal{I}_w \subseteq \mathcal{J}_w$ be the active coordinates of \mathbf{w}^b .

Now suppose $\mathbf{w}^b \succeq \mathbf{w}^c$. We define

$$\mathcal{J} := \bigsqcup_{x \in (\mathcal{Y} \cap \mathcal{Z}) \setminus \mathcal{W}} \mathcal{I}_x \sqcup \bigsqcup_{w \in \mathcal{Y} \cap \mathcal{Z} \cap \mathcal{W}_+} \mathcal{I}_w \sqcup \bigsqcup_{w \in \mathcal{Y} \cap \mathcal{Z} \cap \mathcal{W}_-} \mathcal{I}_w.$$

Then $\mathbf{w}_{\mathcal{J}}^b = \mathbf{w}_{\mathcal{J}}^c$, by assumptions 3, 4 and 5. Meanwhile $\mathbf{w}_{\mathcal{J}}^{a+} = \mathbf{o}_{\mathcal{J}} = \mathbf{w}_{\mathcal{J}}^{a-}$, by assumptions 4 and 5. On the other hand, if $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$, then we have $\mathbf{w}_{\mathcal{K}}^b = \mathbf{w}_{\mathcal{K}}^{a+}$ (by assumptions 1, 4, and 5) and $\mathbf{w}_{\mathcal{K}}^c = \mathbf{w}_{\mathcal{K}}^{a-}$ (by assumptions 2, 4, and 5). Thus, the separability of (\succeq) implies that $(\mathbf{w}^b \succeq \mathbf{w}^c) \iff (\mathbf{w}^{a+} \succeq \mathbf{w}^{a-})$, as desired. \diamond claim 1

Let $\mathcal{C}_+ := \{a \in \mathcal{A}; \mathbf{w}^{a+} \succ \mathbf{w}^{a-}\}$, $\mathcal{C}_- := \{a \in \mathcal{A}; \mathbf{w}^{a+} \prec \mathbf{w}^{a-}\}$, and $\mathcal{C}_0 := \{a \in \mathcal{A}; \mathbf{w}^{a+} \approx \mathbf{w}^{a-}\}$. Then $\mathcal{A} = \mathcal{C}_- \sqcup \mathcal{C}_0 \sqcup \mathcal{C}_+$. Let $\mathcal{C}_{0+} := \mathcal{C}_0 \sqcup \mathcal{C}_+$ and $\mathcal{C}_{0-} := \mathcal{C}_0 \sqcup \mathcal{C}_-$.

Claim 2: *Let $b, c \in \mathcal{A}$.*

- (a) *If $b, c \in \mathcal{C}_{0+}$, then $b + c \in \mathcal{C}_{0+}$. If also $b \in \mathcal{C}_+$ or $c \in \mathcal{C}_+$ then $b + c \in \mathcal{C}_+$.*
- (b) *If $b, c \in \mathcal{C}_{0-}$, then $b + c \in \mathcal{C}_{0-}$. If also $b \in \mathcal{C}_-$ or $c \in \mathcal{C}_-$ then $b + c \in \mathcal{C}_-$.*
- (c) *$b \in \mathcal{C}_{0+}$ if and only if $-b \in \mathcal{C}_{0-}$.*
- (d) *\mathcal{C}_0 is a subgroup of \mathcal{A} .*

Proof. (a) Claim 1(a) says that $b = b_+ - b_-$ and $c = c_+ - c_-$ for some disjoint $b_+, b_-, c_+, c_- \in \mathcal{B}$. Clearly, $b + c = (b_+ + c_+) - (b_- + c_-)$, and $(b_+ + c_+)$ and $(b_- + c_-)$ are also elements of \mathcal{B} .

Define $\mathcal{J}_b^+ := \mathcal{I}(\mathbf{w}^{b+}, \mathbf{o})$, $\mathcal{J}_b^- := \mathcal{I}(\mathbf{w}^{b-}, \mathbf{o})$, $\mathcal{J}_c^+ := \mathcal{I}(\mathbf{w}^{c+}, \mathbf{o})$, and $\mathcal{J}_c^- := \mathcal{I}(\mathbf{w}^{c-}, \mathbf{o})$. By Π_{in} -invariance, we can assume without loss of generality that:

- \mathcal{J}_b^+ , \mathcal{J}_b^- , \mathcal{J}_c^+ , and \mathcal{J}_c^- are disjoint.
- $\mathcal{I}(\mathbf{w}^{b_++c_+}, \mathbf{o}) = \mathcal{J}_b^+ \sqcup \mathcal{J}_c^+$. Furthermore, $\mathbf{w}_{\mathcal{J}_b^+}^{b_++c_+} = \mathbf{w}_{\mathcal{J}_b^+}^{b_+}$ and $\mathbf{w}_{\mathcal{J}_c^+}^{b_++c_+} = \mathbf{w}_{\mathcal{J}_c^+}^{c_+}$.
- $\mathcal{I}(\mathbf{w}^{b_--c_-}, \mathbf{o}) = \mathcal{J}_b^- \sqcup \mathcal{J}_c^-$. Furthermore, $\mathbf{w}_{\mathcal{J}_b^-}^{b_--c_-} = \mathbf{w}_{\mathcal{J}_b^-}^{b_-}$, and $\mathbf{w}_{\mathcal{J}_c^-}^{b_--c_-} = \mathbf{w}_{\mathcal{J}_c^-}^{c_-}$.

Let $\mathcal{J} := \mathcal{I} \setminus (\mathcal{J}_b^+ \sqcup \mathcal{J}_b^- \sqcup \mathcal{J}_c^+ \sqcup \mathcal{J}_c^-)$. Then we can indicate the three assumptions above with the following equations:

$$\begin{aligned} \mathbf{w}^{b_+} &= (\mathbf{w}_{\mathcal{J}_b^+}^{b_+} \quad \mathbf{o}_{\mathcal{J}_b^-} \quad \mathbf{o}_{\mathcal{J}_c^+} \quad \mathbf{o}_{\mathcal{J}_c^-} \quad \mathbf{o}_{\mathcal{J}}); \\ \mathbf{w}^{b_-} &= (\mathbf{o}_{\mathcal{J}_b^+} \quad \mathbf{w}_{\mathcal{J}_b^-}^{b_-} \quad \mathbf{o}_{\mathcal{J}_c^+} \quad \mathbf{o}_{\mathcal{J}_c^-} \quad \mathbf{o}_{\mathcal{J}}); \\ \mathbf{w}^{c_+} &= (\mathbf{o}_{\mathcal{J}_b^+} \quad \mathbf{o}_{\mathcal{J}_b^-} \quad \mathbf{w}_{\mathcal{J}_c^+}^{c_+} \quad \mathbf{o}_{\mathcal{J}_c^-} \quad \mathbf{o}_{\mathcal{J}}); \\ \mathbf{w}^{c_-} &= (\mathbf{o}_{\mathcal{J}_b^+} \quad \mathbf{o}_{\mathcal{J}_b^-} \quad \mathbf{o}_{\mathcal{J}_c^+} \quad \mathbf{w}_{\mathcal{J}_c^-}^{c_-} \quad \mathbf{o}_{\mathcal{J}}); \\ \mathbf{w}^{b_++c_+} &= (\mathbf{w}_{\mathcal{J}_b^+}^{b_++c_+} \quad \mathbf{o}_{\mathcal{J}_b^-} \quad \mathbf{w}_{\mathcal{J}_c^+}^{c_+} \quad \mathbf{o}_{\mathcal{J}_c^-} \quad \mathbf{o}_{\mathcal{J}}); \\ \text{and } \mathbf{w}^{b_--c_-} &= (\mathbf{o}_{\mathcal{J}_b^+} \quad \mathbf{w}_{\mathcal{J}_b^-}^{b_--c_-} \quad \mathbf{o}_{\mathcal{J}_c^+} \quad \mathbf{w}_{\mathcal{J}_c^-}^{c_-} \quad \mathbf{o}_{\mathcal{J}}). \end{aligned} \tag{6}$$

Now, $b \in \mathcal{C}_{0+}$, so $\mathbf{w}^{b_+} \succeq \mathbf{w}^{b_-}$. Applying separability in the \mathcal{J}_c^+ coordinates to the first two equations of (6), we get

$$(\mathbf{w}_{\mathcal{J}_b^+}^{b_++c_+}, \mathbf{o}_{\mathcal{J}_b^-}, \mathbf{w}_{\mathcal{J}_c^+}^{c_+}, \mathbf{o}_{\mathcal{J}_c^-}, \mathbf{o}_{\mathcal{J}}) \succeq (\mathbf{o}_{\mathcal{J}_b^+}, \mathbf{w}_{\mathcal{J}_b^-}^{b_--c_-}, \mathbf{w}_{\mathcal{J}_c^+}^{c_+}, \mathbf{o}_{\mathcal{J}_c^-}, \mathbf{o}_{\mathcal{J}}). \tag{7}$$

Also, $c \in \mathcal{C}_{0+}$, so $\mathbf{w}^{c+} \succeq \mathbf{w}^{c-}$. Applying separability in the \mathcal{J}_b^- coordinates to the third and fourth equations of (6), we get

$$(\mathbf{o}_{\mathcal{J}_b^+}, \mathbf{w}_{\mathcal{J}_b^-}^{b-}, \mathbf{w}_{\mathcal{J}_c^+}^{c+}, \mathbf{o}_{\mathcal{J}_c^+}, \mathbf{o}_{\mathcal{J}}) \succeq (\mathbf{o}_{\mathcal{J}_b^+}, \mathbf{w}_{\mathcal{J}_b^-}^{b-}, \mathbf{o}_{\mathcal{J}_c^+}, \mathbf{w}_{\mathcal{J}_c^+}^{c-}, \mathbf{o}_{\mathcal{J}}). \quad (8)$$

Combining equations (7) and (8) via transitivity, we get

$$(\mathbf{w}_{\mathcal{J}_b^+}^{b+}, \mathbf{o}_{\mathcal{J}_b^-}, \mathbf{w}_{\mathcal{J}_c^+}^{c+}, \mathbf{o}_{\mathcal{J}_c^-}, \mathbf{o}_{\mathcal{J}}) \succeq (\mathbf{o}_{\mathcal{J}_b^+}, \mathbf{w}_{\mathcal{J}_b^-}^{b-}, \mathbf{o}_{\mathcal{J}_c^+}, \mathbf{w}_{\mathcal{J}_c^+}^{c-}, \mathbf{o}_{\mathcal{J}}). \quad (9)$$

Now, matching the two sides of equation (9) with the last two equations in (6), we get $\mathbf{w}^{b_++c_+} \succeq \mathbf{w}^{b_--c_-}$. But $b+c = (b_+ + c_+) - (b_- + c_-)$. Thus, Claim 1(b) implies that $b+c \in \mathcal{C}_{0+}$.

If $b \in \mathcal{C}_+$, then $\mathbf{w}^{b_+} \succ \mathbf{w}^{b_-}$, which makes equation (7) strict, which makes equation (9) strict, which means that $\mathbf{w}^{b_++c_+} \succ \mathbf{w}^{b_--c_-}$, and hence $b+c \in \mathcal{C}_+$. Likewise, if $c \in \mathcal{C}_+$, then equations (8) and (9) become strict, so that $b+c \in \mathcal{C}_+$.

- (b) Similar to (a).
- (c) Let $b \in \mathcal{C}_{0+}$, and write $b = b_+ - b_-$. If $c = -b$, then $c = b_- - b_+$, and these elements are disjoint, so the uniqueness part of Claim 1(a) implies that $c_+ = b_-$ and $c_- = b_+$. We have $\mathbf{w}^{b_+} \succeq \mathbf{w}^{b_-}$ (because $b \in \mathcal{C}_{0+}$); thus, $\mathbf{w}^{c_-} \succeq \mathbf{w}^{c_+}$, so $c \in \mathcal{C}_{0-}$.
- By identical argument, if $-b \in \mathcal{C}_{0-}$, then $b \in \mathcal{C}_{0+}$.
- (d) First note that (a) and (b) together imply that \mathcal{C}_0 is closed under addition, and (c) implies that \mathcal{C}_0 is closed under inverses. Finally, we have $0 \in \mathcal{C}_0$, because $0_+ = 0_- = 0$, so that $\mathbf{w}^{0_-} = \mathbf{w}^{0_+} = \mathbf{w}^0 = \mathbf{o}$. ◇ Claim 2

Define $\mathcal{R} := \mathcal{A}/\mathcal{C}_0$; then \mathcal{R} is an abelian group. Let $\phi : \mathcal{A} \rightarrow \mathcal{R}$ be the quotient map. Then define $\mathcal{R}_+ := \phi(\mathcal{C}_+)$ and $\mathcal{R}_- := \phi(\mathcal{C}_-)$.

Claim 3: (a) For all nonzero $r \in \mathcal{R}$, either $r \in \mathcal{R}_+$ or $-r \in \mathcal{R}_+$, but not both.

(b) For all $r, s \in \mathcal{R}_+$, we have $(r+s) \in \mathcal{R}_+$.

Proof. Mapping Claim 2(a) through ϕ immediately yields (b).

To check (a), note that $\mathcal{A} = \mathcal{C}_- \sqcup \mathcal{C}_0 \sqcup \mathcal{C}_+$; thus, $\mathcal{R} = \phi(\mathcal{C}_-) \cup \phi(\mathcal{C}_0) \cup \phi(\mathcal{C}_+) = \mathcal{R}_- \cup \{0\} \cup \mathcal{R}_+$. Thus, any nonzero element of \mathcal{R} is either in \mathcal{R}_+ or \mathcal{R}_- .

It remains only to show that \mathcal{R}_+ and \mathcal{R}_- are disjoint. By contradiction, suppose $r \in \mathcal{R}_+ \cap \mathcal{R}_-$. Find $b \in \mathcal{C}_+$ and $c \in \mathcal{C}_-$ such that $\phi(b) = \phi(c) = r$. Claim 2(c) implies that $-c \in \mathcal{C}_+$. Thus, Claim 2(a) yields $b-c \in \mathcal{C}_+$. But $\phi(b-c) = \phi(b) - \phi(c) = r - r = 0$, so $b-c \in \mathcal{C}_0$. Contradiction. ◇ Claim 3

Now define a binary relation ($>$) on \mathcal{R} by setting $r > s$ if and only if $r - s \in \mathcal{R}_+$. Claim 3(a) implies that ($>$) is complete and antisymmetric. Claim 3(b) implies that ($>$) is transitive. Thus ($>$) is a total order relation.

Finally, define $u : \mathcal{X} \rightarrow \mathcal{R}$ by $u(x) := \phi(1 \cdot x)$ (where $1 \cdot x$ denotes an element of \mathcal{A}). It remains to show that (\succeq) satisfies statement (2). Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, with $d(\mathbf{x}, \mathbf{y}) < \infty$. Let

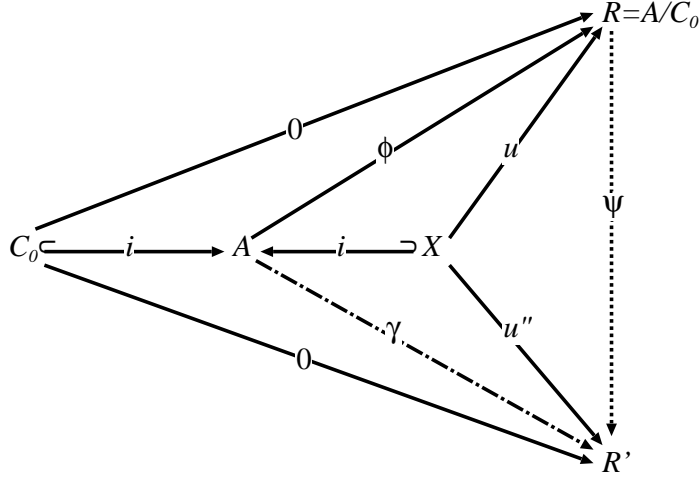


Figure 1: A commuting diagram illustrating the proof of the universal property in Theorem 1. Here i represents an inclusion map (regard \mathcal{X} as a subset of \mathcal{A} in the obvious way) and 0 represents a zero homomorphism.

$\mathcal{K} := \mathcal{I}(\mathbf{x}, \mathbf{y})$ and $\mathcal{J} := \mathcal{I} \setminus \mathcal{K}$, and define $\mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ by setting $\mathbf{x}'_{\mathcal{K}} := \mathbf{x}_{\mathcal{K}}$ and $\mathbf{y}'_{\mathcal{K}} := \mathbf{y}_{\mathcal{K}}$, while $x'_j = y'_j = o$ for all $j \in \mathcal{J}$.

Let x_1, \dots, x_N be the distinct elements of $\mathcal{X} \setminus \{o\}$ which occur in \mathbf{x}' , and for each $n \in [1 \dots N]$, let $J_n \in \mathbb{N}$ be the number of times we see x_n . If $a := J_1 x_1 + \dots + J_N x_N \in \mathcal{B}$, then there exists some $\pi \in \Pi_{\text{fin}}$ such that $\pi(\mathbf{x}') = \mathbf{w}^a$; thus $\mathbf{x}' \approx \mathbf{w}^a$ by Π_{fin} -invariance.

Likewise, let y_1, \dots, y_M be the distinct elements of $\mathcal{X} \setminus \{o\}$ which occur in \mathbf{y}' , and for each $m \in [1 \dots M]$, let $K_m \in \mathbb{N}$ be the number of times we see y_m . If $b := K_1 y_1 + \dots + K_M y_M \in \mathcal{B}$, then there exists some $\tau \in \Pi_{\text{fin}}$ such that $\tau(\mathbf{y}') = \mathbf{w}^b$; thus $\mathbf{y}' \approx \mathbf{w}^b$ by Π_{fin} -invariance. Now we have:

$$\begin{aligned} (\mathbf{x} \succeq \mathbf{y}) &\stackrel{(*)}{\iff} (\mathbf{x}' \succeq \mathbf{y}') \stackrel{(\dagger)}{\iff} (\mathbf{w}^a \succeq \mathbf{w}^b) \stackrel{(\diamond)}{\iff} (a - b \in \mathcal{C}_{0+}) \\ &\stackrel{(\@)}{\iff} (\phi(a - b) \in \mathcal{R}_+ \sqcup \{0\}) \stackrel{(\ddagger)}{\iff} (\phi(a - b) \geq 0). \end{aligned} \quad (10)$$

(*) is by separability, because $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}'$ satisfy the separability conditions (3) by construction. Next, (†) is because $\mathbf{x}' \approx \mathbf{w}^a$ and $\mathbf{y}' \approx \mathbf{w}^b$. Finally, (◇) is by Claim 1(b) and the definition of \mathcal{C}_{0+} , (@) is by definition of \mathcal{R}_+ , and (‡) is by definition of (\succ). Now,

$$\begin{aligned} \phi(a - b) &= \phi(J_1 x_1 + \dots + J_N x_N) - \phi(K_1 y_1 + \dots + K_M y_M) \stackrel{(*)}{=} \sum_{n=1}^N J_n u(x_n) - \sum_{m=1}^M K_m u(y_m) \\ &= \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(x_i) - \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(y_i) = \sum_{i \in \mathcal{I}} (u(x_i) - u(y_i)). \end{aligned} \quad (11)$$

Here (*) is because $\phi(1 \cdot x_n) = u(x_n)$ and $\phi(1 \cdot y_m) = u(y_m)$. Combining statements (10) and (11) yields (2). Thus, (\succeq) is the additive preorder on $\mathcal{X}^{\mathcal{I}}$ defined by u .

Universal property. Let $(\mathcal{R}', +, >)$ be another linearly ordered abelian group, and let $u' : \mathcal{X} \rightarrow \mathcal{R}'$ be some function such that (\succeq) is also the additive preorder defined by u' . Let $r' := u'(o)$, and define $u'' : \mathcal{X} \rightarrow \mathcal{R}'$ by $u''(x) := u'(x) - r'$. Thus $u''(o) = 0$, and (\succeq) is also the additive preorder defined by u'' .

Define $\gamma : \mathcal{A} \rightarrow \mathcal{R}'$ by setting $\gamma(\sum_{w \in \mathcal{W}} A_w w) = \sum_{w \in \mathcal{W}} A_w u''(w)$ for any finite $\mathcal{W} \subseteq \mathcal{X} \setminus \{o\}$ and coefficients $\{A_w\}_{w \in \mathcal{W}} \subseteq \mathbb{Z}$. Equivalently, $\gamma(a) := \sum_{\mathcal{I}} (u''(\mathbf{w}^{a+}) - u''(\mathbf{w}^{a-}))$ for all $a \in \mathcal{A}$. This is automatically a group homomorphism, because \mathcal{A} is the free abelian group generated by $\mathcal{X} \setminus \{o\}$.

Claim 4: $\mathcal{C}_0 \subseteq \ker(\gamma)$.

Proof. If $a \in \mathcal{C}_0$, then $\mathbf{w}^{a+} \approx \mathbf{w}^{a-}$. Thus,

$$\gamma(a) := \sum_{\mathcal{I}} (u''(\mathbf{w}^{a+}) - u''(\mathbf{w}^{a-})) \stackrel{(*)}{=} 0.$$

Thus, $a \in \ker(\gamma)$, as desired. Here $(*)$ is by eqn.(2), because $\mathbf{w}^{a+} \approx \mathbf{w}^{a-}$, and (\succeq) is the additive preorder defined by u'' . ◇ Claim 4

Claim 4 means that γ factors through ϕ to yield a homomorphism $\psi : \mathcal{R} \rightarrow \mathcal{R}'$ such that $\psi \circ \phi = \gamma$. (See Figure 1.)

Claim 5: $u'' = \psi \circ u$.

Proof. Let $x \in \mathcal{X}$. Let $a = 1x$ (an element of \mathcal{A}). Then $u''(x) = \gamma(a) = \psi \circ \phi(a) = \psi(u(x))$. ◇ Claim 5

It remains only to show that ψ is order-preserving. Let $r \in \mathcal{R}$; then $r = \phi(a)$ for some $a \in \mathcal{A}$.

$$\begin{aligned} (r \geq 0) &\iff (a \in \mathcal{C}_{0+}) \iff (\mathbf{w}^{a+} \succeq \mathbf{w}^{a-}) \stackrel{(*)}{\iff} \left(\sum_{\mathcal{I}} (u''(\mathbf{w}^{a+}) - u''(\mathbf{w}^{a-})) \geq 0 \right) \\ &\stackrel{(\dagger)}{\iff} (\gamma(a) \geq 0) \stackrel{(\diamond)}{\iff} (\psi(r) \geq 0), \end{aligned}$$

as desired. Here, $(*)$ is by eqn.(2), because (\succeq) is the additive preorder defined by u'' . Next, (\dagger) is by definition of γ , and (\diamond) is because $\psi \circ \phi = \gamma$ and $\phi(a) = r$. □

Remark. The proof of the ‘existence’ part of Theorem 1 is somewhat analogous to the proof strategy used by Wakker [58]. Let $\mathcal{Y} := \mathcal{X} \setminus \{o\}$. Wakker begins with a separable, anonymous preference order (\succeq) defined on all sequences in $\mathcal{X}^{\mathbb{N}}$ which have only finitely many nonzero entries. He introduces a set $[\mathcal{Y}]$, which is essentially the free abelian monoid generated by \mathcal{Y} (denoted by \mathcal{B}' in our proof). The order (\succeq) induces a complete preorder on $[\mathcal{Y}]$, which is compatible with the addition operation. Wakker also assumes (\succeq) is Archimedean; he then invokes Theorem 3.2.2.1 of Krantz *et al.* [40] to obtain a real-valued, additive utility representation for (\succeq) , equivalent to our Corollary 5.

Proof of Corollary 5. Let $(\mathcal{R}, +, >)$ be a linearly ordered abelian group, let $u : \mathcal{X} \rightarrow \mathcal{R}$, and let (\succeq_u) be the additive preorder defined by (2). Assume without loss of generality that the image set $u(\mathcal{X})$ generates \mathcal{R} (otherwise replace \mathcal{R} with the subgroup generated by $u(\mathcal{X})$.)

Claim 1: (\succeq_u) is Archimedean if and only if \mathcal{R} is Archimedean.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, with $d(\mathbf{x}, \mathbf{o}) < \infty$ and $d(\mathbf{y}, \mathbf{o}) < \infty$. Suppose

$$r = \sum_{i \in \mathcal{I}} (u(x_i) - u(o)) \quad \text{and} \quad s = \sum_{i \in \mathcal{I}} (u(y_i) - u(o)). \quad (12)$$

For any $N \in \mathbb{N}$, define $\mathbf{x}^N \in \mathcal{X}^{\mathcal{I}}$ as in §2. Then

$$\sum_{i \in \mathcal{I}} (u(x_i^N) - u(y_i)) = N \cdot \sum_{i \in \mathcal{I}} (u(x_i) - u(o)) - \sum_{i \in \mathcal{I}} (u(y_i) - u(o)) = N \cdot r - s.$$

Thus, $\mathbf{x}^N \succ \mathbf{y}$ if and only if $N \cdot r - s > 0$.

“ \Leftarrow ” If $\mathbf{x} \succ \mathbf{o}$, then $r > 0$. Since \mathcal{R} is Archimedean, there exists $N \in \mathbb{N}$ such that $N \cdot r > s$, and thus, $\mathbf{x}^N \succ \mathbf{y}$.

“ \Rightarrow ” For any $r, s \in \mathcal{R}$, we can construct some $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ satisfying (12) (because $u(\mathcal{X})$ generates \mathcal{R} by hypothesis). Since (\succeq_u) is Archimedean, there exists $N \in \mathbb{N}$ such that $\mathbf{x}^N \succ \mathbf{y}$ and thus, $N \cdot r > s$. ◇ Claim 1

Now combining Theorem 1, Hölder’s Theorem, and Claim 1 yields the result. □

We will now prove the results of §3, culminating in the proof of Theorem 2. First we define some convenient notation. For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and any function $u : \mathcal{X} \rightarrow \mathcal{R}$, we define ${}^*\sum_{\mathcal{I}} u(\mathbf{x}) := {}^*\sum_{i \in \mathcal{I}} u(x_i)$. Also, for any $\mathcal{F} \in \mathcal{F}$, define $\sum_{\mathcal{F}} u(\mathbf{x}) := \sum_{f \in \mathcal{F}} u(x_f)$.

Proof of Lemma 6. Let $0^{\mathcal{F}}$ be the constant 0 element of $\mathcal{R}^{\mathcal{F}}$, and define $\mathcal{Z} := \{r \in \mathcal{R}^{\mathcal{F}}; r \underset{\mathfrak{U}\mathfrak{F}}{\approx} 0^{\mathcal{F}}\}$;

Claim 1: \mathcal{Z} is a subgroup in $\mathcal{R}^{\mathcal{F}}$.

Proof. For any $r \in \mathcal{R}^{\mathcal{F}}$, let $\mathcal{O}(r) := \{\mathcal{F} \in \mathcal{F}; r(\mathcal{F}) = 0\}$. Then $r \in \mathcal{Z}$ if and only if $\mathcal{O}(r) \in \mathfrak{U}\mathfrak{F}$. But, for any $r, s \in \mathcal{R}^{\mathcal{F}}$, we have $\mathcal{O}(r - s) \supseteq \mathcal{O}(r) \cap \mathcal{O}(s)$; thus, if $r \in \mathcal{Z}$ and $s \in \mathcal{Z}$, then axioms (F1) and (F2) imply $r - s \in \mathcal{Z}$. ◇ Claim 1

For any $r, s \in \mathcal{R}^{\mathcal{F}}$, it is easy to check that $r \underset{\mathfrak{U}\mathfrak{F}}{\approx} s$ if and only if $(r - s) \in \mathcal{Z}$. Thus, ${}^*\mathcal{R}$ is just the quotient group $\mathcal{R}^{\mathcal{F}}/\mathcal{Z}$. The relation $(>)$ defines a linear order on ${}^*\mathcal{R}$. □

Proof of Lemma 9. Definition (4) says that the function $\mathcal{X}^{\mathcal{I}} \ni \mathbf{x} \mapsto {}^*\sum_{\mathcal{I}} u(\mathbf{x}) \in {}^*\mathcal{R}$ is a utility function for the preorder $(\overset{*}{\succeq}_u)$. The value of ${}^*\sum_{\mathcal{I}} u(\mathbf{x})$ is well-defined for all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, and ${}^*\mathcal{R}$ is totally ordered, so $(\overset{*}{\succeq}_u)$ is a complete preorder on $\mathcal{X}^{\mathcal{I}}$.

Separable. Let $\mathcal{J} \subset \mathcal{I}$ and let $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ satisfy the separability conditions (3). We must show that $(\mathbf{x} \stackrel{*}{\sim}_u \mathbf{y}) \Leftrightarrow (\mathbf{x}' \stackrel{*}{\sim}_u \mathbf{y}')$.

Define $\mathcal{F}(\mathbf{x}, \mathbf{y}) := \{\mathcal{F} \in \mathcal{F}; \sum_{\mathcal{F}} u(\mathbf{x}) \geq \sum_{\mathcal{F}} u(\mathbf{y})\}$. Then $(\sum_{\mathcal{I}} u(\mathbf{x}) \geq \sum_{\mathcal{I}} u(\mathbf{y})) \Leftrightarrow (\mathcal{F}(\mathbf{x}, \mathbf{y}) \in \mathfrak{U}\mathfrak{F})$ and $(\sum_{\mathcal{I}} u(\mathbf{x}') \geq \sum_{\mathcal{I}} u(\mathbf{y}')) \Leftrightarrow (\mathcal{F}(\mathbf{x}', \mathbf{y}') \in \mathfrak{U}\mathfrak{F})$. Thus, it suffices to show that $(\mathcal{F}(\mathbf{x}, \mathbf{y}) \in \mathfrak{U}\mathfrak{F}) \Leftrightarrow (\mathcal{F}(\mathbf{x}', \mathbf{y}') \in \mathfrak{U}\mathfrak{F})$. In fact, we will show that $\mathcal{F}(\mathbf{x}', \mathbf{y}') = \mathcal{F}(\mathbf{x}, \mathbf{y})$.

For any $\mathcal{F} \in \mathcal{F}$, the equations (3) imply:

$$\begin{aligned} \text{(a)} \quad \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{x}) &= \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{y}), & \text{(b)} \quad \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{x}) &= \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{x}'), \\ \text{(c)} \quad \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{x}') &= \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{y}'), & \text{and} \quad \text{(d)} \quad \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{y}) &= \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{y}'). \end{aligned}$$

Furthermore, $\mathcal{F} = (\mathcal{F} \cap \mathcal{J}) \sqcup (\mathcal{F} \cap \mathcal{K})$ (because $\mathcal{I} := \mathcal{J} \sqcup \mathcal{K}$); thus

$$\begin{aligned} \text{(e)} \quad \sum_{\mathcal{F}} u(\mathbf{x}) &= \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{x}) + \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{x}), & \sum_{\mathcal{F}} u(\mathbf{y}) &= \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{y}) + \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{y}); \\ \text{(f)} \quad \sum_{\mathcal{F}} u(\mathbf{x}') &= \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{x}') + \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{x}'), & \sum_{\mathcal{F}} u(\mathbf{y}') &= \sum_{\mathcal{F} \cap \mathcal{J}} u(\mathbf{y}') + \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{y}'). \end{aligned}$$

Thus, for all $\mathcal{F} \in \mathcal{F}$, we have:

$$\begin{aligned} (\mathcal{F} \in \mathcal{F}(\mathbf{x}, \mathbf{y})) &\iff (\sum_{\mathcal{F}} u(\mathbf{x}) \geq \sum_{\mathcal{F}} u(\mathbf{y})) \iff_{(*)} (\sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{x}) \geq \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{y})) \\ &\iff_{(\dagger)} (\sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{x}') \geq \sum_{\mathcal{F} \cap \mathcal{K}} u(\mathbf{y}')) \iff_{(\diamond)} (\sum_{\mathcal{F}} u(\mathbf{x}') \geq \sum_{\mathcal{F}} u(\mathbf{y}')) \\ &\iff (\mathcal{F} \in \mathcal{F}(\mathbf{x}', \mathbf{y}')). \end{aligned}$$

Thus, $\mathcal{F}(\mathbf{x}', \mathbf{y}') = \mathcal{F}(\mathbf{x}, \mathbf{y})$. Here, (*) is by equations (a) and (e); (†) is by equations (b) and (d); and (◇) is by equations (c) and (f). \square

Proof of Lemma 10. (a) If $\epsilon \in \Pi$ is the identity permutation, then clearly $\mathcal{F}(\epsilon) = \mathcal{F} \in \mathfrak{U}\mathfrak{F}$. Also, for any $\pi \in \Pi$, it is clear that $\mathcal{F}(\pi^{-1}) = \mathcal{F}(\pi)$, so $\pi \in \Pi_{\mathfrak{U}\mathfrak{F}}$ if and only if $\pi^{-1} \in \Pi_{\mathfrak{U}\mathfrak{F}}$.

Finally, let $\pi_1, \pi_2 \in \Pi_{\mathfrak{U}\mathfrak{F}}$. Then $\mathcal{F}(\pi_1 \circ \pi_2) \supseteq \mathcal{F}(\pi_1) \cap \mathcal{F}(\pi_2)$. But $\mathcal{F}(\pi_1) \cap \mathcal{F}(\pi_2) \in \mathfrak{U}\mathfrak{F}$ by Axiom (F1); thus, $\mathcal{F}(\pi_1 \circ \pi_2) \in \mathfrak{U}\mathfrak{F}$ by (F2); thus, $\pi_1 \circ \pi_2 \in \Pi_{\mathfrak{U}\mathfrak{F}}$.

(b) Let $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi_{\mathfrak{U}\mathfrak{F}}$. Then for every $\mathcal{F} \in \mathcal{F}$, we have $\sum_{\mathcal{F}} u(\mathbf{x}) = \sum_{\pi(\mathcal{F})} u[\pi(\mathbf{x})]$. But if $\mathcal{F} \in \mathcal{F}(\pi)$, then $\pi(\mathcal{F}) = \mathcal{F}$, so we get $\sum_{\mathcal{F}} u(\mathbf{x}) = \sum_{\mathcal{F}} u[\pi(\mathbf{x})]$. If $\mathcal{F}(\pi) \in \mathfrak{U}\mathfrak{F}$, then this implies that $\sum_{\mathcal{I}} u(\mathbf{x}) = \sum_{\mathcal{I}} u[\pi(\mathbf{x})]$; thus, $\mathbf{x} \stackrel{*}{\sim}_u \pi(\mathbf{x})$. \square

Proof of Lemma 12. (a) Recall that $\mathfrak{F}_{\Gamma} := \{\mathcal{E} \subseteq \mathcal{F}; \mathcal{F}_{\Delta}(\mathcal{J}) \subseteq \mathcal{E} \text{ for some finite } \mathcal{J} \subset \mathcal{I} \text{ and finite } \Delta \subseteq \Gamma\}$.

Claim 1: \mathfrak{F} is a free filter.

Proof. We must check axioms (F0)-(F2).

(F0) For any finite subset $\Delta \subseteq \Gamma$, let \mathcal{O}_Δ be the orbit partition generated by $\langle \Delta \rangle$. Then \mathcal{O}_Δ has an infinite number of elements, because \mathcal{I} is infinite, whereas each element of \mathcal{O}_Δ is a finite subset (because Γ has locally finite orbits).

For any finite subset $\mathcal{J} \subseteq \mathcal{I}$ let $\mathcal{O}_\Delta(\mathcal{J}) := \{\mathcal{O}; \mathcal{O} \in \mathcal{O}_\Delta \text{ and } \mathcal{J} \cap \mathcal{O} \neq \emptyset\}$; then $\mathcal{O}_\Delta(\mathcal{J})$ is finite, and $\mathcal{F}_\Delta(\mathcal{J}) := \{\bigsqcup_{\mathcal{P} \in \mathcal{P}} \mathcal{P}; \mathcal{P} \subseteq \mathcal{O}_\Delta \text{ any finite subset such that } \mathcal{O}_\Delta(\mathcal{J}) \subseteq \mathcal{P}\}$. Thus $\mathcal{F}_\Delta(\mathcal{J})$ is infinite, because $\mathcal{O}_\Delta \setminus \mathcal{O}_\Delta(\mathcal{J})$ is infinite.

Thus, if $\mathcal{E} \subseteq \mathcal{F}$ is finite, then we cannot have $\mathcal{F}_\Delta(\mathcal{J}) \subseteq \mathcal{E}$ for any finite subsets $\mathcal{J} \subseteq \mathcal{I}$ and $\Delta \subseteq \Gamma$; thus, $\mathcal{E} \notin \mathfrak{F}$.

(F1) Let $\mathcal{D}, \mathcal{E} \in \mathfrak{F}$. Then there exist finite subsets $\mathcal{J}, \mathcal{K} \subset \mathcal{I}$ and $\Delta, \mathbf{E} \subseteq \Gamma$ such that $\mathcal{F}_\Delta(\mathcal{J}) \subseteq \mathcal{D}$ and $\mathcal{F}_{\mathbf{E}}(\mathcal{K}) \subseteq \mathcal{E}$. Thus, $\mathcal{J} \cup \mathcal{K}$ and $\Delta \cup \mathbf{E}$ are also finite, and we have $\mathcal{F}_{\Delta \cup \mathbf{E}}(\mathcal{J} \cup \mathcal{K}) = \mathcal{F}_\Delta(\mathcal{J}) \cap \mathcal{F}_{\mathbf{E}}(\mathcal{K}) \subseteq \mathcal{D} \cap \mathcal{E}$. Thus, $\mathcal{D} \cap \mathcal{E} \in \mathfrak{F}$ also.

(F2) Let $\mathcal{D} \in \mathfrak{F}$; then $\mathcal{F}_\Delta(\mathcal{J}) \subseteq \mathcal{D}$ for some finite subsets $\mathcal{J} \subset \mathcal{I}$ and $\Delta \subseteq \Gamma$. Thus, for any $\mathcal{E} \subseteq \mathcal{F}$, if $\mathcal{D} \subseteq \mathcal{E}$, then $\mathcal{F}_\Delta(\mathcal{J}) \subseteq \mathcal{E}$ also, so $\mathcal{E} \in \mathfrak{F}$. \diamond Claim 1

Now apply the Ultrafilter Lemma to obtain some ultrafilter $\mathfrak{U}\mathfrak{F}$ which contains \mathfrak{F}_Γ .

(b) Let $\pi \in \Pi_{\text{fin}}$; then $\mathcal{I}(\pi)$ is finite. Thus, for any finite subset $\Delta \subseteq \Gamma$, we have $\mathcal{F}_\Delta[\mathcal{I}(\pi)] \in \mathfrak{F}_\Gamma \subseteq \mathfrak{U}\mathfrak{F}$. But $\mathcal{F}_\Delta[\mathcal{I}(\pi)] \subseteq \mathcal{F}(\pi)$; thus, axiom (F2) implies $\mathcal{F}(\pi) \in \mathfrak{U}\mathfrak{F}$, so $\pi \in \Pi_{\mathfrak{U}\mathfrak{F}}$.

Now let $\gamma \in \Gamma$, and fix a finite $\mathcal{J} \subseteq \mathcal{I}$. If $\mathcal{F} \in \mathcal{F}_{\{\gamma\}}(\mathcal{J})$, then $\gamma(\mathcal{F}) = \mathcal{F}$, so $\mathcal{F} \in \mathcal{F}(\gamma)$. Thus, $\mathcal{F}_{\{\gamma\}}(\mathcal{J}) \subseteq \mathcal{F}(\gamma)$, so $\mathcal{F}(\gamma) \in \mathfrak{F}_\Gamma$, and thus $\mathcal{F}(\gamma) \in \mathfrak{U}\mathfrak{F}$. Thus, $\gamma \in \Pi_{\mathfrak{U}\mathfrak{F}}$. \square

Proof of Proposition 14. Fix some finite $\Delta \subseteq \Gamma$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^\mathcal{I}$ with $d(\mathbf{x}, \mathbf{y}) < \infty$. The set $\mathcal{I}(\mathbf{x}, \mathbf{y})$ is finite, and for any $\mathcal{J} \in \mathcal{F}_\Delta[\mathcal{I}(\mathbf{x}, \mathbf{y})]$, we have:

$$\sum_{\mathcal{J}} u(\mathbf{x}) - \sum_{\mathcal{J}} u(\mathbf{y}) = \sum_{j \in \mathcal{J}} u(x_j) - \sum_{j \in \mathcal{J}} u(y_j) = \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(x_i) - \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(y_i),$$

because $x_j = y_j$ for all $j \in \mathcal{J} \setminus \mathcal{I}(\mathbf{x}, \mathbf{y})$.

If $\mathbf{x} \stackrel{*}{\succeq}_u \mathbf{y}$, then $\sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(x_i) \geq \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(y_i)$; thus, $\sum_{\mathcal{J}} u(\mathbf{x}) \geq \sum_{\mathcal{J}} u(\mathbf{y})$ for all $\mathcal{J} \in \mathcal{F}_\Delta[\mathcal{I}(\mathbf{x}, \mathbf{y})]$. But $\mathcal{F}_\Delta[\mathcal{I}(\mathbf{x}, \mathbf{y})] \in \mathfrak{F}_\Gamma \subseteq \mathfrak{U}\mathfrak{F}$; thus, ${}^*\sum_{\mathcal{I}} u(\mathbf{x}) \geq {}^*\sum_{\mathcal{I}} u(\mathbf{y})$. Thus $\mathbf{x} \stackrel{*}{\succeq}_u \mathbf{y}$.

Likewise, if $\mathbf{x} \stackrel{*}{\succ}_u \mathbf{y}$, then $\sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(x_i) > \sum_{i \in \mathcal{I}(\mathbf{x}, \mathbf{y})} u(y_i)$; thus, $\sum_{\mathcal{J}} u(\mathbf{x}) > \sum_{\mathcal{J}} u(\mathbf{y})$ for all $\mathcal{J} \in \mathcal{F}_\Delta[\mathcal{I}(\mathbf{x}, \mathbf{y})] \in \mathfrak{U}\mathfrak{F}$; thus, ${}^*\sum_{\mathcal{I}} u(\mathbf{x}) > {}^*\sum_{\mathcal{I}} u(\mathbf{y})$. Thus $\mathbf{x} \stackrel{*}{\succ}_u \mathbf{y}$. \square

Proof of Lemma 15.

Claim 1: Let $\mathcal{G} \in \mathfrak{U}\mathfrak{F}$.

(a) $\mathbf{x} \stackrel{*}{\succeq}_{u, \mathcal{G}} \mathbf{y}$ if and only if $\sum_{\mathcal{J}} u(\mathbf{x}) \geq \sum_{\mathcal{J}} u(\mathbf{y})$ for all $\mathcal{J} \in \mathcal{G}$.

(b) $\mathbf{x} \stackrel{*}{\succ}_{u, \mathcal{G}} \mathbf{y}$ if and only if $\sum_{\mathcal{J}} u(\mathbf{x}) > \sum_{\mathcal{J}} u(\mathbf{y})$ for all $\mathcal{J} \in \mathcal{G}$.

Proof. Fix $\mathcal{J} \in \mathcal{G}$ and $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, and let $\mathbf{x}' := \mathbf{x}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}}$ and $\mathbf{y}' := \mathbf{y}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}}$. Then $\mathcal{I}(\mathbf{x}', \mathbf{y}') \subseteq \mathcal{J}$, so $d(\mathbf{x}', \mathbf{y}') < \infty$ (because \mathcal{J} is finite). Thus, the (\succeq_u) -order of \mathbf{x}' and \mathbf{y}' is well-defined, and Proposition 14 says that

$$\left(\mathbf{x}' \succeq_u \mathbf{y}'\right) \iff \left(\mathbf{x}' \overset{*}{\succeq}_u \mathbf{y}'\right) \quad \text{and} \quad \left(\mathbf{x}' \succ_u \mathbf{y}'\right) \iff \left(\mathbf{x}' \overset{*}{\succ}_u \mathbf{y}'\right). \quad (13)$$

$$\text{Furthermore, } \sum_{i \in \mathcal{I}} \left(u(x'_i) - u(y'_i)\right) = \sum_{\mathcal{J}} u(\mathbf{x}) - \sum_{\mathcal{J}} u(\mathbf{y}). \quad (14)$$

(a) “ \implies ” Suppose $\mathbf{x} \overset{*}{\succeq}_u \mathbf{y}$. Then for any $\mathcal{J} \in \mathcal{G}$ and $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, if \mathbf{x}' and \mathbf{y}' are defined as above, then $\mathbf{x}' \overset{*}{\succeq}_u \mathbf{y}'$. Thus, statement (13) says that $\mathbf{x}' \succeq_u \mathbf{y}'$, so $\sum_{i \in \mathcal{I}} \left(u(x'_i) - u(y'_i)\right) \geq 0$. But then statement (14) implies that $\sum_{\mathcal{J}} u(\mathbf{x}) \geq \sum_{\mathcal{J}} u(\mathbf{y})$.

“ \impliedby ” Fix $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and $\mathcal{J} \in \mathcal{G}$. If \mathbf{x}' and \mathbf{y}' are defined as above, then statement (14) implies that $\sum_{i \in \mathcal{I}} \left(u(x'_i) - u(y'_i)\right) \geq 0$, so $\mathbf{x}' \succeq_u \mathbf{y}'$, so statement (13) says $\mathbf{x}' \overset{*}{\succeq}_u \mathbf{y}'$.

This holds for all $\mathcal{J} \in \mathcal{G}$ and $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$; thus, $\mathbf{x} \overset{*}{\succeq}_{u, \mathcal{G}} \mathbf{y}$.

(b) The proof is similar to (a); change “ \geq ” to “ $>$ ” and “ \succeq ” to “ \succ ” everywhere. \diamond **claim 1**

(a) “ \implies ” Suppose $\mathbf{x} \overset{*}{\succeq}_u \mathbf{y}$. Thus, if $\mathcal{G} := \{\mathcal{F} \in \mathcal{F}; \sum_{\mathcal{F}} u(\mathbf{x}) \geq \sum_{\mathcal{F}} u(\mathbf{y})\}$, then $\mathcal{G} \in \mathcal{U}\mathfrak{F}$. By definition, $\sum_{\mathcal{J}} u(\mathbf{x}) \geq \sum_{\mathcal{J}} u(\mathbf{y})$ for all $\mathcal{J} \in \mathcal{G}$. Thus, Claim 1(a) says that $\mathbf{x} \overset{*}{\succeq}_{u, \mathcal{G}} \mathbf{y}$.

“ \impliedby ” If $\mathbf{x} \overset{*}{\succeq}_{u, \mathcal{G}} \mathbf{y}$; then Claim 1(a) says that $\sum_{\mathcal{J}} u(\mathbf{x}) \geq \sum_{\mathcal{J}} u(\mathbf{y})$ for all $\mathcal{J} \in \mathcal{G}$. But $\mathcal{G} \in \mathcal{U}\mathfrak{F}$, so this means that ${}^*\sum_{\mathcal{I}} u(\mathbf{x}) \geq {}^*\sum_{\mathcal{I}} u(\mathbf{y})$, which means $\mathbf{x} \overset{*}{\succeq}_u \mathbf{y}$.

(b) The proof is the same as (a), but using Claim 1(b) instead of Claim 1(a). \square

Proof of Lemma 16. “ \impliedby ” follows from Proposition 14 and Lemma 15.

“ \implies ” Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, and suppose $\mathbf{x} \overset{*}{\succeq}_u \mathbf{y}$. Then Lemma 15(a) yields some $\mathcal{G} \in \mathcal{U}\mathfrak{F}$ such that, for all $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and $\mathcal{J} \in \mathcal{G}$, we have $\mathbf{x}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}} \overset{*}{\succeq}_u \mathbf{y}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}}$, and thus, $\mathbf{x}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}} \succeq_u \mathbf{y}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}}$, by Proposition 14. But then $\mathbf{x}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}} \succeq_{\text{fin}} \mathbf{y}_{\mathcal{J}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}}$, because $(\succeq_{\text{fin}}) = (\succeq_u)$ by hypothesis.

This holds for all $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and $\mathcal{J} \in \mathcal{G}$; thus, (C1) forces $\mathbf{x} \succeq \mathbf{y}$, because (\succeq) is $\mathcal{U}\mathfrak{F}$ -continuous. Thus, we have $(\mathbf{x} \overset{*}{\succeq}_u \mathbf{y}) \implies (\mathbf{x} \succeq \mathbf{y})$.

Likewise, if $\mathbf{x} \overset{*}{\succ}_u \mathbf{y}$, then Lemma 15(b) and (C2) imply that $\mathbf{x} \succ \mathbf{y}$.

Lemma 9 says $(\overset{*}{\succeq}_u)$ is a complete preorder on $\mathcal{X}^{\mathcal{I}}$. It follows that $(\succeq) = (\overset{*}{\succeq}_u)$. \square

Proof of Theorem 2. “ \impliedby ” follows from Lemmas 9, 10(b), 12(b), and 15.

“ \implies ” Let (\succeq_{fin}) be the finitary part of (\succeq) . Then (\succeq_{fin}) is Π_{fin} -invariant and separable, so Theorem 1 yields a linearly ordered abelian group $(\mathcal{R}, +, >)$ and $u : \mathcal{X} \rightarrow \mathcal{R}$ such that $(\succeq_{\text{fin}}) = (\succeq_u)$. Then Lemma 16 implies that $(\succeq) = (\overset{*}{\succeq}_u)$.

The existence of an \mathcal{R} and u with the universal property follows from the construction in Theorem 1. \square

Proof of Proposition 17. Fix $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{F}_x := \{\mathcal{F} \in \mathcal{F}; \sum_{\mathcal{F}} u(\mathbf{x}) \geq \sum_{\mathcal{F}} u(\mathbf{y})\}$, while $\mathcal{F}_y := \mathcal{F}_x^c = \{\mathcal{F} \in \mathcal{F}; \sum_{\mathcal{F}} u(\mathbf{x}) < \sum_{\mathcal{F}} u(\mathbf{y})\}$.

“ \implies ” If $\mathbf{x} \stackrel{\Gamma}{\succsim}_u^* \mathbf{y}$, then there exists some $\mathcal{J} \in \mathcal{F}$ and finite $\Delta \subset \Gamma$ such that $\sum_{\mathcal{F}} u(\mathbf{x}) \geq \sum_{\mathcal{F}} u(\mathbf{y})$ for all $\mathcal{F} \in \mathcal{F}_{\Delta}(\mathcal{J})$. Thus, $\mathcal{F}_{\Delta}(\mathcal{J}) \subseteq \mathcal{F}_x$. But $\mathcal{F}_{\Delta}(\mathcal{J}) \in \mathcal{U}\mathfrak{F}$ for any Γ -admissible free ultrafilter $\mathcal{U}\mathfrak{F}$; thus, $\mathcal{F}_x \in \mathcal{U}\mathfrak{F}$ by (F2). Thus, $\mathbf{x} \stackrel{*}{\succsim}_u^{\mathcal{U}\mathfrak{F}} \mathbf{y}$.

“ \impliedby ” (by contrapositive) Suppose $\mathbf{x} \not\stackrel{\Gamma}{\succsim}_u^* \mathbf{y}$. Then there is no $\mathcal{J} \in \mathcal{F}$ and finite $\Delta \subseteq \Gamma$ with $\mathcal{F}_{\Delta}(\mathcal{J}) \subseteq \mathcal{F}_x$. Thus, $\mathcal{F}_{\Delta}(\mathcal{J}) \cap \mathcal{F}_y \neq \emptyset$ for every $\mathcal{J} \in \mathcal{F}$ and finite $\Delta \subseteq \Gamma$. Thus,

$$\mathcal{E} \cap \mathcal{F}_y \neq \emptyset, \quad \text{for every } \mathcal{E} \in \mathfrak{F}_{\Gamma}. \quad (15)$$

Now define $\mathfrak{F}_{\Gamma}^y := \{\mathcal{D} \subseteq \mathcal{F}; \mathcal{E} \cap \mathcal{F}_y \subseteq \mathcal{D} \text{ for some } \mathcal{E} \in \mathfrak{F}_{\Gamma}\}$.

Claim 1: (a) \mathfrak{F}_{Γ}^y is a free filter, and (b) $\mathfrak{F}_{\Gamma} \subseteq \mathfrak{F}_{\Gamma}^y$.

Proof. (a) \mathfrak{F}_{Γ}^y satisfies (F1) because \mathfrak{F}_{Γ} satisfies (F1) and (15). It satisfies (F2) by construction. To verify (F0), it suffices to show that $\mathcal{E} \cap \mathcal{F}_y$ is infinite for any $\mathcal{E} \in \mathfrak{F}_{\Gamma}$.

By contradiction, suppose $\mathcal{E} \cap \mathcal{F}_y$ is finite. Each element of $\mathcal{E} \cap \mathcal{F}_y$ is a finite subset of \mathcal{I} . Let $\mathcal{F} := \bigcup \{\mathcal{E}; \mathcal{E} \in \mathcal{E} \cap \mathcal{F}_y\}$; then \mathcal{F} is also finite, hence, a proper subset of \mathcal{I} . Let $\mathcal{D} \subset \mathcal{I}$ be a nonempty finite subset disjoint from \mathcal{F} ; then no element of $\mathcal{E} \cap \mathcal{F}_y$ contains \mathcal{D} . Thus, for any finite $\Delta \subseteq \Gamma$, we have $\mathcal{F}_{\Delta}(\mathcal{D}) \cap \mathcal{E} \cap \mathcal{F}_y = \emptyset$. But $\mathcal{F}_{\Delta}(\mathcal{D}) \cap \mathcal{E} \in \mathfrak{F}_{\Gamma}$ by (F2), because $\mathcal{F}_{\Delta}(\mathcal{D}) \in \mathfrak{F}_{\Gamma}$ by definition. This contradicts (15). By contradiction, $\mathcal{E} \cap \mathcal{F}_y$ must be infinite.

(b) Let $\mathcal{E} \in \mathfrak{F}_{\Gamma}$. Then $\mathcal{E} \supseteq \mathcal{E} \cap \mathcal{F}_y$; hence $\mathcal{E} \in \mathfrak{F}_{\Gamma}^y$ by definition. ◇ Claim 1

Claim 1(a) and the Ultrafilter Lemma says there exists a free ultrafilter $\mathcal{U}\mathfrak{F}^y$ containing \mathfrak{F}_{Γ}^y . Claim 1(b) implies that $\mathcal{U}\mathfrak{F}^y$ also contains \mathfrak{F}_{Γ} , so it is Γ -admissible.

Claim 2: $\mathbf{x} \stackrel{*}{\succsim}_u^{\mathcal{U}\mathfrak{F}^y} \mathbf{y}$.

Proof. Let $\mathcal{E} \in \mathfrak{F}_{\Gamma}$ be arbitrary. Then $\mathcal{E} \cap \mathcal{F}_y \in \mathfrak{F}_{\Gamma}^y$ by definition, and $\mathcal{E} \cap \mathcal{F}_y \subseteq \mathcal{F}_y$, so $\mathcal{F}_y \in \mathfrak{F}_{\Gamma}^y$ by (F2), so $\mathcal{F}_y \in \mathcal{U}\mathfrak{F}^y$, which means $\mathbf{x} \stackrel{*}{\succsim}_u^{\mathcal{U}\mathfrak{F}^y} \mathbf{y}$. ◇ Claim 2

Thus, it is false that $\mathbf{x} \not\stackrel{*}{\succsim}_u^{\mathcal{U}\mathfrak{F}} \mathbf{y}$ for every Γ -admissible ultrafilter $\mathcal{U}\mathfrak{F}$ on \mathcal{F} . □

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